Introduction

This article is based on a talk the author gave at the 4th meeting on "History and Technology in Math Education" in Rio de Janeiro, Brazil. I want to summarize here the past and present of computer technology in school teaching, discuss hopes and problems, and try to imagine the implications looking ahead into the near and far future.

Parallel to the general discussion, the article looks at the specific program C.a.R. (Compass and Ruler, Régua e Compasso) written by the author, which is one of the many available dynamic geometry tools. Let us look ahead into the future development of this program.

Computer technology has changed our lives in more ways than we realize at this moment, and it will continue to change our world with a tremendous momentum. While everybody can see that computers have become an integral part of the world around us, the implications of the exponential development of the world wide web are still not fully understood by most of us.

At this moment, the web is starting to be more than just a collection of information pieces, vaguely held together by search engines. The web starts to be the place where the work is done, and where files are stored. The client system no longer plays the major role it used to play in the past. Information is no longer sold on CDs and "installed" on a user computer, but available in and through the web. Likewise, light web applications take the role of large monolithic software suites. Consequently, the way software should be designed, and hardware should be chosen will have to change.

In the following, let us discuss the hopes and fears arising from this development, and weather it is applicable to schools at all. Let us try to imagine the role the computer can play in schools of tomorrow.

Dynamic Geometry

To advertise the role a computer program could play in teaching, look at the sample program C.a.R. (Compass and Ruler) developed by the author of this article.

The program uses an approach to teach geometric content, which is best described by "dynamic geometry". This term is, however, trademarked by Geometer's sketchpad, a similar program to teach or demonstrate geometric content. There are numerous programs of this type with all levels of sophistication. A few of them are quite advanced, being easy and versatile to use, and feature rich. C.a.R. has been developed in Java in the 90ies with predecessors in C++ for Windows, OS/2 and DOS. The first dynamic programs appeared in the 80ies. Cabri was probably the forerunner of this idea.

Dynamic geometry means that geometric constructions can be studied by moving one or more of the basic points. A construction can be an ordinary compass and ruler construction, or can be very advanced even involving numerical solutions to complicated equations. For purposes of basic education in geometry, simple compass and ruler constructions, or simple functions and curves given by expressions, will do.

To show the basic ideas, we discuss the middle perpendicular of a line as shown in the image.

Of course, we can use the program the stupid way applying a macro to get the middle perpendicular of AB. This will work, but it does not teach anything.
Seemingly more advanced is the construction on the left. It is however, only a construction, and does not have mathematical content either, unless we add an explanation. The construction itself is just a procedure. Kids may like this procedure and may want to play with it, but it is nevertheless an empty recipe to follow.

Mathematics starts, when we observe (or define) that the middle perpendicular is the line of points with equal distance to A and B. We can construct this line if we have two points on it. From the above construction, it is obvious that the two green points have just the property we are asking for. They have equal distance to A and B. We can find these points with two simple circles. Hence, we can find the middle perpendicular.

But why is the locus of points with equal distance to A and B a line?

Asking this question is where mathematical reasoning starts. The author of these lines is not advocating to get into this type of questions at each and every mathematical concept. However, as a professor of mathematics, he would be glad if a bit more thinking found its way into mathematical education, replacing the stupid repetition and mechanical practicing of procedures.

On the other hand, we must not go back to the books of Euclid, which were the base of mathematical teaching for over a thousand years. It is not the rigorous proof I am asking for, but the education for curiousness and experimentation, the education to ask good questions and find answers. I.e., we like to follow the following outline.

- understand the problem,
- ask the precise question,
- experiment with it,
- conjecture a solution,
- proof the solution.

To demonstrate this, let us give a more advanced example than the middle perpendicular.

Since we are already aware that the middle perpendicular is the locus of all points with equal distance to A and B, we understand what a locus is. For the following we pose ourselves the question: What is the locus of points having twice the distance to A than to B? In formula, this reads as $2PB=PA$ (not the other way around, which is a well known mistake).

To experiment with this question, we use dynamic geometry. We try to find such a point P on an arbitrary ray through A. Of course, one could display the distances PA and PB, or even the quotient PA/PB, in the construction window of a program like C.a.R., and then play around with P to get this ratio close to 2. But it is the first challenge in this problem to actually construct the point P.

We do this with a "similarity construction". Take any point H on the ray, find the middle I of HA, and finally a point on AB with $HA=2*HJ$. We then blow up the triangle AHJ to the similar triangle APB using a parallel to HJ through B. This gives us our point P.

In a dynamic geometry program, we can now vary the green ray, and observe what happens to the point P. If we print out the ration PA/PB in the construction window, this will convince us that our construction of P is actually correct.

As a side remark, we observe that sometimes there is no solution. If the angle is too large, the circle HI will not intersect AB. Moreover, we see that most of the time there are two solutions.
In the image above, we added the second solution $P'$ (using J'), and generated the tracks of $P$ and $P'$ while we moved the green point Q which was used to define the ray AQ. Surprisingly, we come to the conjecture that the locus we are looking for is a circle!

Once we know it is a circle, we observe that the circle must be symmetric to the line AB. Thus its center is on the line AB. Moreover, we can compute two points $P$ and $P'$ on AB, which satisfy our equation. So we can construct the circle. This is another challenge that can be solved by careful observation and thinking, combined with a bit of computation. To construct the circle involves dividing AB into three parts, by the way, which is another challenge.

The deepest, and most difficult part of this all starts, when we try to proof that the solution is a circle. I am not asking for a rigorous proof in terms of Euclid or Hilbert, but for anything, which convinces us that the circle is the solution. After all, this fact is not obvious! It is perfectly in the spirit of axiomatic mathematics, to base a proof on any previous knowledge available. If there is no or not much previous knowledge, there can be no proof. Proofs based on high level knowledge are sometimes referred to as "local proofs", in contrast to proofs that base on profound axioms.

Now, that I convinced you that the proof might be interesting, I like to show one of the ideas that could be used. Another idea will be given in the next section.

For any M, we can set the radius MP in the figure on the right in such a way, that $MP/BM=AM/MP$. If we do this, the triangles MPA and MBP will be similar. Thus $PA/PB$ will be constant. If we choose M properly, it will be 2.

Of course, if we ask for the locus such that $PA-PB=2$, we get a hyperbola, which is probably a new object never met before. Thus, experiments leads to new objects, which leads to new questions, and we are on a path to real mathematics.

**Computer Algebra**

Much time is wasted in the classroom with the training of routines like the manipulation of large fractions, tedious algebraic transforms, or the basic techniques of calculus. Since time is a limited resource, it is tempting to use a computer algebra system (CAS), as soon as the basic manipulation procedures are know to the learner. The time regained could be used to climb into more real life, and consequently more difficult examples, into mathematical thinking and understanding as indicated in the previous section.

Examples of CAS software are Maple, Mathematica and Derive. Derive is built in pocket computers made by Texas Instruments combined with a dynamic geometry system, and Casio has a CAS in a pocket computer too. The problems arising from this kind of equipment have to be discussed below. There is the free CAS system Maxima with various graphical interfaces. One of them is Euler, written by the author of this paper. It combines fast numerical computations with computer algebra. Below is a screen print.
However, we should be aware that the ability to compute in one’s head, or on paper, is a basic cultural skill. And it is easily observed, that these skills are lost, once a computer or a pocket computer is available. The degree of this loss is indeed frightening. However, the author strictly claims that the gains are higher. As we do no longer teach calligraphy outside of art classes, or no longer teach how to take the square root on paper, we can stop wasting time with longish and boring computations beyond the basic acquaintance needed in everyday life.

Of course, all this assumes that the gained time is well spent! The syllabus, as well as the tests, must reflect the new thinking. Some change can be observed since PISA showed us “open problems”. An open problem is a problem, which does not have a clear procedure and path to the solution. Such problems not only can be solved in various ways according to the skills and knowledge of the learner, but also can lead to new questions.

I like to point out, that the geometry problem of the last chapter can also be solved with algebra. Computer algebra is not really helpful here, since it takes more time to figure out how to do the manipulations in the CAS, than it takes to do the manipulations by hand. If we assume $A=(-1,0)$ and $B=(1,0)$, we get for $P=(x,y)$

$$(x + 1)^2 + y^2 = 2((x - 1)^2 + y^2) \iff x^2 - 6x + 1 + y^2 = 0 \iff (x - 3)^2 + y^2 = 8$$

Note that computer algebra is not always the easiest way to go. Try to prove that the point $Q$ on the image at the right runs on an ellipse. Using algebraic expressions in some straightforward way leads to nasty equations.

However, there is a simple and elegant geometric proof. If you look carefully, you find out that the ellipse is the circle of radius $c+d$ distorted by the factor $d/(c+d)$ along the $y$-axis. It is an easy challenge to see this in a geometric way.

**Other Software for Schools**

Of course, software for schools is not only restricted to mathematics software, or hardware in form of pocket computers. It is obvious how text editing can be used in schools to allow nice looking texts, journals, or other material. Likewise, presentations are richer if they are generated with the help of a computer. As an example, function
plots can be inserted into documents to show mathematical functions of one, two or even three dimensions.

We have to discuss computer based learning and training here. This is often called "self guided learning" to cover the fact that the student is on his own when judging when and where to proceed. Only very few system for very few topics can currently be called successful. In most cases, not even vocabulary training falls into that category, since vocabulary is best learned in context, which the simple programs are missing. In short, until the programs become very much better in giving feedback and social context, the role of the teacher cannot be replaced by a machine. The hopes of the society to spare money this way are yet in vain.

This is not to say that there cannot be a place for computer based learning content. After all, most of us have successfully learned from books, and computers can very well replace books, and even enrich the material with multimedia. In fact we will later stress the fact that online content will play a major role in future teaching. However, to guide and inspire the learner still takes a human being. A robot replacing a good teacher is science fiction. (A bad teacher, however, can be replaced by anything!)

The other type of software in existence are demo programs. These programs could be of use in any natural science. A simple example is the computer simulation of the solar system with the software of the Celestia project. However, the ideas seem unlimited. A computer can make the invisible visible, and the dangerous possible.

For mathematics, either dynamic geometry, or a simple function plotter can be used to demonstrate otherwise invisible things. A set of functions can be animated, and an expression be computed depending on some parameter. The plotted result can give new insight in the nature of the problem at hand. Both Geogebra and C.a.R. can be used as a demonstration tool of this nature.

One possible path to use such a demo system in mathematics could be

- to demonstrate something with a computer,
- to try out various examples and check the solution, if possible,
- then to raise the question, how the computer did that and if we could do it too,
- then to explain and teach the method.

Even for simple things like solving a quadratic equation this could be a way to approach the problem, and to awaken the curiosity of the children. Then, after they learned how to do it themselves, they still have the computer to check their solution. It is like screwing apart a toy to see how it is built internally, and why its construction functions as it does. To learn how a machine does something and to use it with this knowledge looks better to me, than to try to train the children to act like machines. However, we will later have to discuss the problems teachers are facing with this approach.

The Past

Since the 90ies schools in Germany have been equipped with computer labs. These are special classrooms with rows of computer terminals and screens, and usually a teacher workplace.

These computer labs were used for special computer related classes. Some of the German states introduced computer classes quite early following the idea that computers will play a central role in the future life of all of us. If children will spend most of their working time at computers later, so the basic statement, the school must prepare for this. Usually, these classes contained introduction courses into computer languages like Pascal, or programming with simple languages like logo with its turtle graphics, or introduction into object oriented thinking. The basic critique against this is that time is a
limited resource in schools, and we must carefully select the topics for school teaching. It is questionable if the usually quite huge amount of time spent to learn a computer language is worth the effort.

Later, the computer classes were connected to the Internet to allow research of the children via the World Wide Web. Moreover, one idea at that time was that children could get in contact to other countries and cultures via email contacts.

The observation was that special classrooms take a special effort. Computers cannot be used casually. Instead, computer classes have to be planned in advance, and extra time has to be reserved to change the room, start the computers, and fix eventual problems in the computer lab. Moreover, the physical dimensions of the computer screens splits the class into individual users or small groups. Last, but not least, teacher expertise with computers was very limited. Moreover, didactical concepts were lacking, or at least rather sparse.

To summarize the past was characterized by

• computer labs in special classrooms with internal and later external network connection,
• no, or very few computers at home,
• big effort to hold classes in the lab,
• little computer experience on the side of the kids,
• lack of teacher experience with computers,
• missing didactical concepts,
• expensive hardware and software.

The Present

Now (in 2008) the situation has changed very much. In only 15 years, the computer became a wide spread part of our homes, and of our offices. The Internet plays an ever increasing role in our communication, with its quick email messages, the chat rooms, the social networks, the search engines, and online content like the Wikipedia encyclopedia.

While computers at home are omnipresent, often even with broad band access to the Internet, the situation in the schools has not changed much. Very few schools have a computer available in every classroom, and in most schools only few classrooms are equipped with a fixed beamer. Computers and beamers must still be transported to the classroom, if they are to be used in teaching. Consequently, the computer is rarely used in German schools today.

Of course, our first concern must be what to do with a computer in the classroom! We will discuss this question later in more detail. Here, it suffices to say that the availability of good content and material for teaching is making progress, albeit only slowly. Internet sites driven by the states are slowly filling with material designed for schools. On the other hand, the aid of the computer is not yet taken into consideration in school books. Teacher education is slowly facing the ever growing role of the computer. But in most states, didactical courses in computer teaching are not yet mandatory.

Let us summarize the current situation.

• Computers are now available in almost all homes (98% in Germany) with web access, often broad band access.
• At home, these computers are frequently used for school purposes like word processing, preparation of presentations, Internet research, or training software.
• Also they are frequently used for purposes not related to the school like chats, social networks, email, or games. Assemblies at LAN parties are a typical example.
• Computers are rarely used in the school (30% in Germany with a frequency of at least twice a week).
• Many classrooms are equipped with fixed beamers, or a beamer is available for the classroom.
• But very few classrooms are equipped with a fixed, readily available computer.
• Computer science ("Informatik") is a regular topic in many German states.
• Computer knowledge and technical acquaintance is very good on the side of the kids, but often too sparse on the side of the teacher.
• Younger teachers use servers with teaching material.

Problems and Hopes

In this section, we like to consider the benefits and the problems connected to the use of computer in schools. Teachers confronted with the question why they do not use the computer more in their everyday teaching give the following answers.

• The computer takes too much effort to use. Computers are still available only in computer labs, which means switching the classroom, setting up the computers in the lab, and supervising the individual activities.
• The content given in the syllabus absorbs all available time. In general, teachers have no time to try own concepts, and to enrich the teaching with their own content, let alone to allow for computer based experimental learning.
• Computer related didactical concepts are not integrated into school books.
• The computer or pocket computer leads to a loss of cultural techniques, like computing in the head or with pencil and paper, book reading, or even handwriting.
• Children abuse the computer at home and get distracted too much with games, Internet chat, and surfing. The school needs to be a place to recover from this, and learn to concentrate on educational content.

While these arguments need to be taken seriously, some of them reflect the lack of imagination of the role the computer will play in our future society. And some are simply based on the current insufficient equipment and aids we are facing in current school life. Nevertheless, all these arguments are valid in schools today.

What are the hopes we connect with computers and the world wide web in schools? Some of them I mentioned in the introductory examples about a new mathematical teaching. But let us take a broader view.

• The computer could be used for experimentation and research. It does for mathematics what classical labs do for natural sciences. And even for natural sciences, it can do things that cannot be done in a laboratory. It can make invisible things visible.
• The computer could be used for scientific and social communication. It allows quick exchange of written messages, or other multimedia content. We can even imagine video conferences across continents, which might allow social or scientific cooperation across cultures. In some large countries, computer communication and distance teaching is a necessity.
• The communication technology associated with computers and the web allows to spread open content prepared by teachers for teachers, or by learners for learners. Previously, only strong companies had the resources to distribute content widely. Now, everyone can do this.

If we think of the uncensored way the web allows to distribute opinions, we see that computers are not only useful for mathematics. In fact, even social sciences could benefit from Internet research.

However, before striving into such vague ideas, let us stick with mathematics and natural sciences. On these areas, we are hoping for more experimental learning, which tries to awaken the curiosity and thus a deeper interest in the subject. Moreover, we are hoping for more real life problems, which might even be of open nature. We are hoping for less procedure routine, and more thinking.

Finally, a very important point is that, whatever we do with the computer, we must take care and pay attention! We have to educate the children to a critical use of the computer.
and the web. Internet resources are not reliable, especially not Wikipedia. Web content may be outright wrong, or at least misleading. This educational aim alone is a good argument not to neglect the computer and the web in schools.

Commercial Influences

With the computers there come companies. I am not talking of the kind of "branding" that takes place with the operation system or the word processor we use. I am talking about the replacement of education by job training.

First of all, the lack of money the public is willing to pay for the education, is leading to "public private partnerships", which means sponsoring of equipment in schools, or even school buildings. It should be obvious that these partnerships are based on interests, be it only to influence the public education into a direction useful for the next generation of future workers. Of course, education has always been guided by its usefulness for the society. But this should include aims like critical thinking, and political awareness. The aim must be to form a well informed personality capable of independent thinking. This should actually be in the best interest of our economy.

So we need to take care that computers are not only used to teach children how to use them in the office. Presentation software, spreadsheets and word processors are not what I am thinking off primarily when I think of computers in the school. Moreover, these things are actually not worth teaching, since children learn them automatically and with ease outside of the school.

The Future

As a software author, I have to think about the computer of the future. For this, I imagine the following developments.

- Computers will get lighter, more compact, more endurable, and with longer lasting battery supply. Big screens will not longer get in the way of the student and the teacher, and keyboards will be replaced by more versatile touch screens.
- Each computers will be connected at all times. Of course, for classrooms this connection will be controlled by the teacher.
- The application and the operating system will loose its dominant position. Instead, the web will be the place to work. Applications will run on the server, or with minimal client support, and can be used on any operating system. Data will be stored in the web.
- Communication tools will be omnipresent. E.g., home assignments will be delivered through the web or by email.
- Enriched, and quality controlled teaching content will be available in the web.

For all this to happen, we still need a lot of technological progress. During the last 15 years the software has been driven by technology. The dynamical geometry software C.a.R mentioned above looked quite different even 5 years ago. Faster computers allowed for more complex graphical output with anti-aliased lines, transparent objects, and complicated computations of loci. Fast access to the web leads to software updating itself, and to direct help from the web community. If we ever reach a point where the need and not the technological feasibility commands the software development, I do not know.

Intergeo

Finally, as an example of answers we give to the new web centered approach to software development, I like to present the Intergeo projected funded by the European Community. C.a.R. is associated partner in Intergeo.
Intergeo will

- offer **content** in a searchable and metadata-tagged portal.
- enable users to use their software of choice by specifying a common **file format** based on open standards.
- **test** available material in the classroom. All stakeholders, software teams, resource authors, teachers and learners will be involved, in order to promote quality enhancement cycles.

This is one example, where web thinking replaces application thinking. However, it takes huge efforts to create such a common lingua franca for geometry. There are obstacles, which lie in the different approaches of the geometry systems. To be more precise, there are two ways of handling the geometric objects internally.

- The geometric content is stored in a **tree** of objects with each object depending on the other objects.
- The content is a list of things to do, or a list of **constrains** for each object.

The latter form may allow much more complicated instructions. While these involved constructions may not be useful in basic geometric education, they can be necessary for more complicated geometric demos.

**Conclusion**

We definitely cannot prevent technology to creep into the schools. Today, most kids have computers in their home, and we have to deal with that. Moreover, the computer and the net will play a central role in our lives. Of course, we can base our exams on computer free skills, like we used to do for decades, thus discouraging the use of computers for any school related activity. But we should not do so. On the contrary, teachers and teacher trainers should try to take the challenge to form the computer into an educational tool. We may even need to educate about computers to prevent technological and cultural chaos.

Doing this, we should think ahead of what we have now. The current situation will change, and computers will become more available, friendlier to use, and more versatile. The Internet will become central to our activities.

I definitely do not want to change school teaching completely. On the contrary, I believe that only a small correction, and only a little shift into the right direction is needed. For mathematics education, we need a bit more time for teaching by exploring, for own activity, for less teacher guided learning. We need to be a bit more brave to avoid boring and useless routines, and come to more open thinking.