General Topology

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Contents

Chapter 1

Metric Spaces

1.1. Definition: A **metric** on a set X is a mapping

$$(x, y) \mapsto d(x, y) \in \mathbb{R}$$

for $x, y \in X$, with the properties

$$\begin{gathered} d(x,x) = 0 \;\Leftrightarrow\; x = 0 \\ d(x,y) = d(y,x) \\ d(x,z) \leq d(x,y) + d(y,z) \end{gathered}$$

for all $x, y, z \in X$. A set X with a metric d is called a **metric space**.

1.2. Problem: Proof

$$d(x, y) \geq 0$$

for all x, y in a metric space X.

1.3. Problem: Proof that for a normed linear space X with norm $\|\cdot\|$ the mapping

$$d(x, y) = \|x - y\|.$$

is a metric.

1.1 Neighborhoods

1.4. Definition: We define the sets

$$\begin{aligned} B_r(x) &:= \{y \in X : d(x, y) < r\}, \\ D_r(x) &:= \{y \in X : d(x, y) \leq r\}. \end{aligned}$$

in a metric space X, and call these sets open (closed) ball with radius r around x.

1.5. Problem: Proof

$$B_\rho(y) \subseteq B_r(x)$$

for $\rho \leq r - d(x,y)$, and

$$B_\rho(y) \subseteq X \setminus B_r(x)$$

for $\rho \leq d(x,y) - r$. (Note that $B_r(x) = \emptyset$, if $r < 0$.) Proof the same for $D_r(x)$.

1.6. Definition: A set $U \subseteq X$ is a **neighborhood** of x, if there is an $\epsilon > 0$, such that

$$x \in B_\epsilon(x) \subseteq U.$$

The set of all neighborhoods of x is called $\mathcal{U}(x)$. We will later call this the neighborhood filter of x.

1.7. Definition: A set $U \subseteq X$ is **open** in X, if it is a neighborhood of all its points.

1.8. Problem: Proof that $B_r(x)$ and $X \setminus D_r(x)$ are open for all $x \in X$ and $r \geq 0$.

1.2 Equivalent Norms

1.9. Problem: Proof that the metrics d_1 and d_2 on X generated by two equivalent norms $\|\cdot\|_1$ and $\|\cdot\|_2$ have the same neighborhoods. Norms are called **equivalent**, if there are constants $c_1, c_2 > 0$ such that

$$c_1\|x\|_1 \leq \|x\|_2 \leq c_2\|x\|_2$$

for all $x \in X$.

1.10 Theorem: *All norms on a finite dimensional space are equivalent.*

1.3 Examples

1.11. Problem: The mapping

$$d(x,y) = \begin{cases} 0, & x = y, \\ 1, & x \neq y, \end{cases}$$

is a metric on each set X, called the **discrete** metric. We have

$$\mathcal{U}(x) = \{A \subseteq X : x \in A\}$$

Why is this metric never generated by a norm for infinite spaces?

1.12. Problem: If d is a metric on X, show that for all $c > 0$

$$d_c(x,y) = \max\{c,\ d(x,y)\}$$

is also a metric. Show that this metric has the same neighborhoods.

1.13. Problem: Let $\mathcal{A}(X,Y)$ be the set of all mappings from sets X to Y. For a metric space Y, show that for any $c > 0$

$$d(f,g) = \max\left\{c,\ \sup_{x \in X} d(f(x), g(x))\right\}$$

is a metric on $\mathcal{A}(X,Y)$, called the metric of **uniform convergence**. (Note that the supremum in the definition may be ∞.)

1.14. Problem: Let X be a measure space with measure μ. Then

$$d(f,g) = \max\left\{c, \int |f-g|\,d\mu\right\}$$

is a metric on the set of all measurable functions $f : X \to \mathbb{R}$. If we restrict to the set of all integrable functions, the

$$d(f,g) = \int |f-g|\,d\mu$$

is a metric, generated by a norm.

1.4 Convergence

1.15. Definition: A sequence $(x_n)_{n\in\mathbb{N}}$ in a metric space X **converge** to $x \in X$, if $d(x,x_n)$ converges to 0.

1.16. Problem: Equivalently, each neighborhood of x must contain all but finitely many elements of the sequence. So each neighborhood must contain an **end piece**

$$M_n = \{x_k : k \geq n\}$$

of the sequence.

1.17. Definition: $A \subseteq X$ is called **closed**, if the limit of each converging sequence in A is in A.

1.18 Theorem: *$A \subseteq X$ is closed, if and only if $X \setminus A$ is open.*

1.19. Problem: Proof this. Note that you need to apply the axiom of choice for a countable system of sets for one direction of the proof.

1.20. Remark: We write

$$U^c := X \setminus U,$$

for the **complement** of U in X, if it is clear, which set X is the basic set.

1.21. Definition: A point $x \in X$ is called an **accumulation point** of a sequence $(x_n)_{n\in\mathbb{N}}$ in a metric space X, if for each neighborhood $U \in \mathcal{U}(x)$ and each $n \in \mathbb{N}$ there is a $k > n$ with $x_k \in U$.

1.22. Problem: So each neighborhood must contain at least one element in each end piece of the sequence.

1.23. Problem: Proof that x is a limit point of the sequence $(x_n)_{n\in\mathbb{N}}$, if and only if there exists a subsequence converging to x. Note that you need the axiom of choice for a countable system of sets for one direction of the proof.

1.24. Remark: If the first condition of a metric d is weakened to $d(x,x) = 0$, we get a **pseudo-metric**. Then it may happen that $d(x,y) = 0$ for $x \neq y$.

1.25. Problem: Give an example of a pseudometric.

1.5 Complete Metric Spaces

1.26. Definition: A sequence $(x_n)_{n\in\mathbb{N}}$ is a **Cauchy sequence** in a metric space X, if for each $\epsilon > 0$ there is an $N_\epsilon \in \mathbb{N}$ with

$$d(x_n, x_m) < \epsilon$$

for all $n, m > N_\epsilon$. The space is called **complete**, if every Cauchy sequence converges.

1.27. Problem: Proof that a sequence is a Cauchy sequence, if and only if for each $\epsilon > 0$ there is a neighborhood $B_\epsilon(x_\epsilon)$ of some $x_\epsilon \in X$ containing an end piece of the sequence. Thus each converging sequence is a Cauchy sequence.

1.28. Problem: Proof that a Cauchy sequence with a convergent subsequence converges.

1.29. Problem: Show that each sequence with

$$d(x_n, x_{n+1}) \le \frac{1}{2^n}$$

is a Cauchy sequence, and that each Cauchy sequence contains a subsequence of this type. Generalize this for $d(x_n, x_{n+1}) \le a_n$.

1.30. Problem: Proof that for a sequence, which does not contain a Cauchy subsequence, there is an $\epsilon > 0$ and a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ such that

$$d(x_{n_l}, x_{n_m}) > \epsilon$$

for all l, m.

1.31. Problem: Proof that a normed space X is complete (a **Banach** space), if and only if each absolutely convergent series, i.e. each series with

$$\sum_{k=1}^{\infty} \|a_k\| < \infty,$$

converges in X.

1.32 Theorem: **(Banach fixed point Theorem)** *Let X be a complete metric space, and $f : X \to X$ contract, i.e., there is a constant $c < 1$ such that*

$$d(f(x), f(y)) \le c\, d(x, y) \qquad \text{for all } x, y \in X.$$

Then f has a unique fixed point $x \in X$, i.e. a unique point with $f(x) = x$. Furthermore the sequence

$$x_0 \in X, \quad x_{n+1} = f(x_n)$$

converges to the fixed point, and the speed of convergence can be estimated by

$$d(x_n, x) \le \frac{c^n}{1-c}\, d(x_0, x_1).$$

1.33. Problem: Proof this theorem.

1.34. Problem: Show that the sequence $x_{n+1} = \cos(x_n)$ converges for all $x_0 \in \mathbb{R}$.

Chapter 2

Topologies

2.1. Definition: A system of sets $\mathcal{T}$ is called a topology on a set X, if

$$\emptyset \in \mathcal{T}, \quad X \in \mathcal{T},$$
$$X_1, \ldots, X_n \in \mathcal{T} \;\Rightarrow\; X_1 \cap \ldots \cap X_n \in \mathcal{T},$$
$$X_i \in cT,\, i \in I \;\Rightarrow\; \bigcup_{i \in I} X_i \in \mathcal{T}.$$

So finite intersections and any unions of sets in $\mathcal{T}$ must be in $\mathcal{T}$. The sets in $\mathcal{T}$ are called **open** in the topology. Their complements are called **closed** sets. A set with a topology is called a **topological space**.

2.2. Problem: Show that the open sets in a metric space form a topology.

2.1 Neighborhoods

2.3. Definition: For x in a topological space X, we define

$$\mathcal{U}(x) = \{V \subseteq X : x \in U \subseteq V \text{ for some open set } U\}.$$

Each $U \in \mathcal{U}(x)$ is called a **neighborhood** of x.

2.4. Problem: Assure that this is in accordance with the definition of neighborhoods in a metrics space.

2.5. Problem: Show that $U \subseteq X$ is open, if and only if it is a neighborhood of all its elements.

2.2 Examples

2.6. Problem: Let X be an infinite set, and U the set of all **cofinite** sets in X, i.e. the set of all sets with finite complement in X. Show that this is a topology, and describe the closed sets and the neighborhoods. Can this be generated by a metric?

2.7. Problem: The power set $\mathcal{P}(X)$ of all subsets of X clearly is a topology. Show that it is generated by a metric. It the the **finest** (largest) topology in X.

2.8. Problem: The set $\mathcal{T} = \{\emptyset, X\}$ is a topology on X. Show that it is not generated by a metric, if X has at least two points. It is the **coarsest** (smallest) topology in X.

2.9. Problem: For two topologies $\mathcal{T}_1$ and $\mathcal{T}_2$ define

$$\mathcal{T} := \{A \cap B : A \in \mathcal{T}_1, B \in \mathcal{T}_2\}.$$

Show that this is a topology, which is **finer** (larger) than $\mathcal{T}_1$ and $\mathcal{T}_2$.

2.3 Interior, Closure and Boundary

2.10. Definition: The smallest closed set containing A is called the **closure** of A and denoted by $\overline{A}$. The largest open set contained in A is called the **interior** of A and denoted by A°. The set

$$\partial A = \overline{A} \setminus A^\circ$$

is called the **boundary** of A.

2.11. Problem: Proof that the intersection of closed set is again closed. Why does it follow that the closure of a set exists? What about the interior?

2.12. Problem: Proof that $x \in \overline{A}$, if and only if each neighborhood of x contains a point of A, and that $x \in \partial A$, if and only if each neighborhood of x contains a point of A and a point of $X \setminus A$.

2.13. Problem: Is it true for closed sets A that $\overline{A^\circ} = A$, or for open sets A that $(\overline{A})^\circ = A$. Can one prove at least a subset relation?

2.4 Bases

2.14. Definition: The coarsest (smallest) topology containing a system $\mathcal{B}$ of subsets of a set X is called the topology **generated by** $\mathcal{B}$.

2.15. Problem: Proof that the topology generated by the system of sets $\mathcal{B} \subseteq \mathcal{P}(X)$ consists of the sets $\emptyset$, X, and all unions of finite intersections of sets in $\mathcal{B}$.

2.16. Definition: A set of open sets $\mathcal{B} \subset \mathcal{T}$ is called a **basis** of the topology $\mathcal{T}$ on X, if for all $x \in X$ and $U \in \mathcal{U}(X)$ exists a $B \in \mathcal{B}$ with $x \in B \subseteq U$.

2.17. Problem: Proof that each open set is the union of basis elements.

2.18. Problem: Proof that $\mathbb{R}^n$ with the usual topology has a countable basis.

2.19. Problem: Proof that $\mathcal{B} \subseteq \mathcal{P}(X)$ is a basis of the topology generated by it, if and only if each $x \in X$ is contained in some $B \in \mathcal{B}$, and if for each

$x \in B_1 \cap B_2$ with $B_1, B_2 \in \mathcal{B}$, there is a $B \in \mathcal{B}$ with

$$x \in B \subseteq B_1 \cap B_2.$$

E.g., the latter condition is satisfied, if $\mathcal{B}$ is stable with respect to intersections.

2.20. Problem: Proof that the system of sets $[a, b[,\ a, b \in \mathbb{R}$ is a basis of a topology on $\mathbb{R}$, which is finer than the usual topology on $\mathbb{R}$.

2.21. Problem: Proof that the system of sets $]a, \infty[,\ a \in \mathbb{R}$, is a basis of a topology on $\mathbb{R}$, which is coarser than the usual topology on $\mathbb{R}$. If the sets $]-\infty, b[,\ b \in \mathbb{R}$, are added to the basis, we get the usual topology on $\mathbb{R}$.

Chapter 3

Continuous Functions

3.1. Definition: A function $f : X \to Y$ between two topological spaces is called **continuous** in X, if for all $V \in \mathcal{U}(f(x))$ exists an $U \in \mathcal{U}(x)$ with $f(U) \subseteq V$. f is continuous, if it is continuous in all $x \in X$.

3.2. Problem: Proof the that concatenation of two continuous functions is continuous.

3.1 Preimages of Sets

3.3 Theorem: *A mapping $f : X \to Y$ between two topological spaces is continuous, if and only if the preimage $f^{-1}(U)$ of each open set U in Y is open in X.*

3.4. Problem: Proof this. Proof that "‘open"’ can be replaced by "‘closed"’ in the theorem.

3.5. Problem: Proof that it satisfies to check, that the preimages of the sets in a basis of the topology are open.

3.2 Continuity by Limits

3.6. Problem: Let X and Y be metric spaces. Proof that $f : X \to Y$ is continuous in x, if and only if

$$x_n \to x \;\Rightarrow\; f(x_n) \to f(x)$$

for all sequences $(x_n)_{n \in \mathbb{N}}$. Note that you need the axiom of choice for a countable system of sets for one direction of the proof.

3.7. Problem: Define

$$\lim_{t \to x,\, t \neq x} f(t)$$

for a mapping $f : X \to Y$ between topological spaces.

3.3 Examples

3.8. Problem: Study the continuous functions $f : \mathbb{R}_r \to \mathbb{R}$, where $\mathbb{R}_r$ is equipped with the topology generated by $[a, b[$, $a, b \in \mathbb{R}$. Since the topology is finer, we get more continuous functions. Proof that f is continuous, if and only if

$$\lim_{t \to x,\, t > x} f(t) = f(x)$$

for all $x \in X$.

3.9. Problem: Study the continuous functions $f : \mathbb{R} \to \mathbb{R}_r$. This time, we get less continuous functions. In fact, only very few functions are still continuous.

3.10. Problem: Study the continuous functions $f : \mathbb{R}_r \to \mathbb{R}_r$.

3.11. Problem: Study the continuous functions $f : \mathbb{R} \to \mathbb{R}_l$, where $\mathbb{R}_l$ is equipped with the topology generated by $]a, \infty[$, $a \in \mathbb{R}$. Since this topology is coarser, we get more continuous functions. Proof that f is continuous, if and only if

$$x_n \to x \;\Rightarrow\; \liminf f(x_n) \geq f(x)$$

for all sequences $(x_n)_{n \in \mathbb{N}}$ in $\mathbb{R}$. Functions satisfying this are called lower **semi-continous**. Likewise, we define upper semi-continous. A function, which is lower and upper semi-continuous is continuous. Generalize this statement for two topologies on Y and a function $f : X \to Y$.

Chapter 4

Induced Topology

4.1. Definition: Let X_i, $i \in I$, be topological spaces, and

$$f_i : X \to X_i, \qquad i \in I,$$

mappings from some set X to the spaces X_i. Then the coarsest topology on X, such that all f_i are continuous is called the topology **induced** by f_i, $i \in I$.

4.2. Problem: Proof that the systems of sets of the form

$$f_{i_1}^{-1}(U_{i_1}) \cap \ldots \cap f_{i_n}^{-1}(U_{i_n})$$

for a finite set of pairwise different indices $i_1, \ldots, i_n \in I$ and open sets $U_{i_k} \subseteq X_{i_k}$ is a basis of the induced topology.

4.3. Problem: Proof that it suffices to choose each U_i from a given basis of each topology on X_i.

4.1 Product Spaces

4.4. Definition: Let X_i, $i \in I$, be topological spaces. Then the induced topology on the product

$$\prod_{i \in I} X_i = \{(x_i)_{i \in I} : x_i \in X_i\}$$

by the projections $\pi_j : \prod_i X_i \to X_j$ defined by

$$\pi_j((x_i)_{i \in I}) = x_j$$

is called the **product topology**.

4.5. Problem: Draw the basis elements of the product topology in $\mathbb{R}^3$, taking the open intervals $]a, b[$ as a basis of the topology in $\mathbb{R}$.

4.6. Problem: Let $(X_n)_{n \in \mathbb{N}}$ be metric spaces with metrics d_n, $n \in \mathbb{N}$. Proof that the metric

$$d((x_n)_{n\in\mathbb{N}},(y_n)_{n\in\mathbb{N}}) = \sum_{k=1}^{\infty} \max\left\{1/2^k, d_k(x_k,y_k)\right\}$$

is a metric and generates the product topology on $\prod_n X_n$.

4.7. Problem: Let U be open in $X = \prod_i X_i$. Proof that all π_i are open mappings, i.e., they map open sets to open sets.

4.2 Relative Topology

4.8. Definition: Let X be a topological space and $A \subseteq X$. Then the **relative topology** on A is the topology induced by the identity mapping $\mathrm{id} : A \to X$.

4.9. Problem: Show that $U \subseteq A$ is open in A, if and only if, there is an open $V \subseteq X$, such that

$$U = V \cap A.$$

Proof the same for "'closed"' instead of "'open"'.

4.10. Problem: Let $A \subseteq X$ be equipped with the relative topology. Formulate continuity of functions $f : A \to Y$ in terms of the topology on X.

4.3 Continuity in Induced Topologies

4.11 Theorem: *Let X have the topology induced by $f_i : X \to X_i$, $i \in I$, and let T be a topological space. Then a function $f : T \to X$ is continuous, if and only if $f_i \circ f$ is continuous for all $i \in I$.*

Proof: The first direction follows from the fact that concatenations of continuous functions are continuous. Now assume all $f_i \circ f$ are continuous. Take $U \in \mathcal{U}(x)$ for $x \in X$, $x = f(t)$, $t \in T$. We may assume that

$$U = f_{i_1}^{-1}(U_{i_1}) \cap \ldots \cap f_{i_n}^{-1}(U_{i_n})$$

with $U_{i_k} \in \mathcal{U}(f_{i_k}(x))$. So there is a neighborhood $V_{i_k} \in \mathcal{U}(t)$ with

$$f_i(f(V_{i_k})) \subseteq U_{i_k}.$$

Set

$$V = V_{i_1} \cap \ldots \cap V_{i_n}.$$

It remains to prove $f(V) \subseteq U$ (left as a problem). □

4.12. Remark: Especially $f : T \to \prod_i X_i$ is continuous, if and only if all f_i, $i \in I$, are continuous, where

$$f(t) = (f_i(t))_{i\in I}.$$

The functions $f_i = \pi \circ f$ are called the **component** functions of f.

4.13. Problem: Let A be a subspace of the topological space X with the relative topology. Then $f : T \to A$ is continuous, if and only if $f : T \to X$ is continuous.

4.4 Coinduced Topologies

4.14. Definition: Let X_i, $i \in I$, be topological spaces and $f_i : X_i \to X$ mappings. Then the finest topology on X such that all f_i are continuous is called the **coinduced** topology on X.

4.15. Problem: Show that this topology exists, and that its open sets U are characterized by the fact that $f_i^{-1}(U)$ is open for all $i \in I$.

4.16. Example: A prominent example is the **quotient topology**. If "‘$\sim$"’ is an equivalence relation on a topological space X, we can define the projection

$$\pi : X \to X/\sim$$

mapping each $x \in X$ to its equivalence class $[x]_\sim$. π then coinduces the quotient topology on $X/\sim$.

4.17. Problem: For a product space $X = \prod_i X_i$ each projection π_i coinduces a topology $\mathcal{T}_i$ on X. Identify the open sets of these topologies. Prove that the product topology is the topology generated by all these topologies.

Chapter 5

Separation

5.1. Definition: Two points are **indistinguishable** by a topology, if they lie in the same open sets. A topology is called **T0**, if all points are distinguishable.

5.2. Problem: Proof that being indistinguishable is an equivalence relation. The quotient topology of this relation is T0.

5.1 T1 Spaces

5.3. Definition: A topological X is called **T1**, if for each points $x, y \in X$ exists an $U \in \mathcal{U}(y)$ with $x \notin U$.

5.4. Problem: X is T1, if and only if all sets $\{x\}$, $x \in X$, are closed.

5.2 T2 Spaces

5.5. Definition: A topological space X is called **T2** or **Hausdorff**, if for $x, y \in X$ exists $U_x \in \mathcal{U}(x)$ and $U_y \in \mathcal{U}(y)$ with $U_x \cap U_y = \emptyset$. We say, that each two points can be **separated** by open sets.

5.6. Problem: A Hausdorff space is T1.

5.7. Problem: X is Hausdorff, if and only if the **diagonal**

$$D = \{(x, x) \in X \times X : x \in X\}$$

is closed in $X^2 = X \times X$.

5.8. Problem: Proof that if $f : X \to Y$ is continuous and Y is Hausdorff, then **graph** of f

$$G_f := \{(x, f(x)) \in X \times Y : x \in X\}$$

is closed in $X \times Y$. Provide an example, that the contrary is not true.

5.3 Regular and Normal Spaces

5.9. Definition: A topological space X is called **regular**, if each point can be separated from each closed set, not containing the point, by open sets. It is called **normal**, if any two disjoint closed sets can be separated by open sets.

5.10. Remark: A regular T1 space is sometimes called T3, and a normal T1 space is sometimes called T4. However, sometimes these notations are used differently. We will speak of a regular T1, and a normal T1 space to avoid any misunderstanding.

5.11. Problem: Show that two disjoint closed sets in a normal space can actually be separated by closed sets.

5.4 Completely Regular Spaces

5.12. Definition: A topological space X is called **completely** regular, if any point x and a closed set A not containing x can be separated by a continuous function.

5.13. Problem: Proof that a completely regular space is regular.

5.14 Theorem: **(Urysohn)** *In each normal space X two disjoint closed sets A and B can be separated by a continuous function. I.e., there is continuous $f : X \to [0, 1]$, which is 0 on A, and 1 on B.*

Proof: Since the space is normal, we can define open sets U_p for dyadic rationals

$$p = \frac{n}{2^m}, \qquad 0 \le n \le 2^m$$

such that

$$U_p \subseteq \overline{U_q} \subseteq U_q$$

and $A \subseteq U_0$, $U_1 = X \setminus B$. Then the function

$$f(x) = \begin{cases} 1, & x \in B \\ \inf\{r \in [0, 1] : x \in U_r\}, & x \notin B, \end{cases}$$

has the desired properties. To show that it is continuous, show that

$$f^{-1}[0, s[= \bigcup_{t<s} U_t$$

$$f^{-1}]s, 1] = \bigcup_{t>s} (X \setminus \overline{U_t}),$$

and use the fact that these sets are a basis of the topology of $[0, 1]$. □

5.15. Problem: For a metric space X with metric d check that $d(x, y)$ is continuous on $X \times X$. Proof that the minimal distance function

$$d(x, A) = \inf\{d(x, a) : a \in A\}$$

is continuous for any nonempty $A \subseteq X$. If A is closed, then

$$x \in A \Leftrightarrow d(x, A) = 0.$$

For two disjoint closed sets A and B in X, show that the function

$$f(x) = \frac{d(x, A)}{d(x, A) + d(x, B)}$$

is continuous, and separates the closed sets A and B.

5.16 Theorem: **(Tietze)** *Let A be a closed set in the normal space X, and $f : A \to [a, b]$ be continuous. Then f can be extended to a continuous function $f : X \to [a, b]$.*

Proof: We may assume $[a, b] = [-1, 1]$. Set

$$C = f^{-1}[-1, -1/3], \quad D = f^{-1}[1/3, 1].$$

The C and D are closed in A and thus in X. We can separate these sets with a function

$$g : X \to [-1/3, 1/3]$$

such that g is $-1/3$, on C and $1/3$ on D. Now we set

$$f_0 = f, \quad g_0 = g,$$

In the next step, we set

$$f_1 = (f_0 - g_0)|_A$$

Thus

$$f_1(A) \subseteq [-2/3, 2/3].$$

We repeat the step above with f_1 and get a sequence of functions

$$f_n : A \to \left[-\left(\frac{2}{3}\right)^n, \left(\frac{2}{3}\right)^n\right], \quad g_n : X \to \left[-\frac{1}{3}\left(\frac{2}{3}\right)^n, \frac{1}{3}\left(\frac{2}{3}\right)^n\right],$$

with

$$f_{n+1} = (f_n - g_n)|_A$$

so that $g = \sum g_n$ converges on X. Clearly $g(x) = f(x)$ for all $x \in A$. □

5.5 Product Spaces

5.17. Problem: Proof that the product space is Hausdorff, if and only if each space is Hausdorff.

5.18 Theorem: *The product of topological spaces is regular, if and only if each space is regular. The product of spaces is completely regular, if and only if each space is completely regular.*

Proof: If x is a point in the product $X = \prod_i X_i$, and A a closed set not containing x. Then we can find a neighborhood of x of the form

$$U = \pi_{i_1}^{-1}(U_{i_1}) \cap \ldots \cap \pi_{i_n}^{-1}(U_{i_n}) \subseteq X \setminus A$$

with open neighborhoods U_{i_k} of x_{i_k}. If each X_i is regular, we can separate x_{i_k} from $X_{i_k} \setminus U_{i_k}$. Using these separations, we can separate x from A. The proof for the complete regular case is similar and left as a problem, as well as the inverse implications. $\square$

Chapter 6

Compact Sets

6.1. Definition: A topological space X is called **compact**, if every covering of X with open sets contains a finite subcovering.

6.2. Remark: We often add the Hausdorff property. Some authors include it into the definition of compact, and call non-Hausdorff compact sets **precompact**.

6.3. Problem: Describe the compactness of subsets of a topology. These sets are sometimes called **relatively compact**.

6.4. Problem: Proof that X is compact, if and only if each intersection of closed sets with the property that each finite intersection is not empty (finite intersection property) is not empty.

6.5. Problem: Proof that the cofinite topology is compact, as well as every subset of it.

6.6. Problem: Proof that the finite union of compact sets is compact.

6.1 Relative Compact Sets

6.7. Problem: Let X be compact. If $A \subseteq X$ is closed, then it is relatively compact.

6.8. Problem: Let X be Hausdorff. If $A \subseteq X$ is compact, then it is closed.

6.9. Problem: Proof that the intersection of any system of compact sets in a topology is compact.

6.2 Continuous Images of Compact Sets

6.10 Theorem: *The continuous image of a compact topological space is compact.*

6.11. Problem: Proof this theorem using open coverings, and assure that Hausdorff is not needed.

6.12. Problem: Let X be compact. Proof that every continuous $f : X \to \mathbb{R}$ has a Maximum and a Minimum.

6.3 Homemorphisms

6.13 Theorem: *Let X be a compact topological space, Y a Hausdorff topological space, and $f : X \to Y$ continuous and bijective. Then $f^{-1} : Y \to X$ is continuous.*

Proof: Let $U \subseteq X$ be open. We have to show that the preimage $f(U)$ of U under f^{-1} is open in Y. Now $X \setminus U$ is compact, and thus $f(X \setminus U)$. Since Y is Hausdorff, this set is closed in Y. So

$$f(U) = Y \setminus f(X \setminus U)$$

is open in Y (proof of this identity left as problem). □

6.14. Definition: We call a bijective continuous mapping $f : X \to Y$ with continuous inverse mapping an **homeomorphism** between X and Y, and X and Y homeomorphic. A homeomorphism from X to a subspace of Y is called an **embedding** of X in Y.

6.15. Problem: Proof that being homeomorphic is an equivalence relation between topologies.

6.4 Compact Metric Spaces

6.16 Theorem: *A metric space X is compact, if and only if every sequence has an accumulation point (i.e., X is* ***sequence-compact****).*

Proof: "'$\Rightarrow$"': Otherwise there is a sequence $(x_n)_{n\in\mathbb{N}}$, and to each $x \in X$ there is an open neighborhood $U \in \mathcal{U}(x)$, which contains only finitely many elements of the sequence. This open covering cannot have a finite covering.

"'$\Leftarrow$"': Otherwise there is an open covering $\mathcal{M}$ without a finite subcovering. First proof that there is a $\delta > 0$, such that $B_\delta(x) \in U$ for some $U \in \mathcal{M}$ (left as problem). Then we have reduced the problem to an open covering with balls of the same radius. If this does not have an open covering, there is a sequence $(x_n)_{n\in\mathbb{N}}$ with

$$d(x_n, x_m) > \delta$$

for all $n \neq m$ (left as a problem). This sequence cannot have a limit point (left as a problem). Note that you needed the axiom of choice for a countable system of sets for this direction. □

6.17. Problem: Each compact Hausdorff topological space is normal.

6.18. Problem: If the graph of a function is compact, the function is continuous.

6.19 Theorem: *Each compact metric space is complete.*

6.20. Problem: Proof this theorem.

6.5 The Hausdorff Metric

6.21. Problem: Denote by $\mathcal{K}(X)$ the set of all non-empty compact subsets of a metric space X. Define

$$d(A, B) = \max_{a \in A} d(a, B)$$

for $A, B \in \mathcal{K}(X)$ and

$$h(A, B) = \max \{d(A, B), d(B, A)\}.$$

Set

$$A_\epsilon = \{x \in X : d(x, A) < \epsilon\}$$

Thus

$$h(A, B) < \epsilon \;\Leftrightarrow\; A \subseteq B_\epsilon \text{ and } B \subseteq A_\epsilon$$

Show that h is a metric on $\mathcal{K}(X)$, the so called **Hausdorff metric**.

6.22. Problem: Proof for $A, B, C, D \in \mathcal{K}(X)$

$$h(A \cup B, C \cup D) \leq \max\{h(A, C), h(B, D)\}.$$

Use

$$(C \cup D)_\epsilon = C_\epsilon \cup D_\epsilon$$

and the same for A and B.

6.23 Theorem: *X is a complete metric space, if and only if $\mathcal{K}(X)$ (with the Hausdorff metric) is a complete metric space.*

Proof: The reverse direction is left as a problem. Assume $(A_n)_{n \in \mathbb{N}}$ is a Cauchy sequence in $\mathcal{K}(X)$. Define

$$A = \{x \in X : d(x, A_n) \to 0\}.$$

We show $A_n \to A$.

Using previous problems, we may assume

$$h(A_n, A_{n+1}) \leq \frac{1}{2^n}.$$

So it is possible to select a sequence $x_n \in A_n$ with

$$d(x_n, x_{n+1}) \leq \frac{1}{2^n}.$$

This Cauchy sequence converges in X. Thus $A \neq \emptyset$.

If $x \notin A$, then $d(x, A_n) > r$ for some $r > 0$ and all $n \in \mathbb{N}$. Thus $B_r(x)$ is completely contained in the complement of A, and A is closed (details as problem).

Let $(x_n)_{n\in\mathbb{N}}$ be any sequence in A. Take $\epsilon > 0$ and $N \in \mathbb{N}$ such that $h(A_N, A_m) < \epsilon$ for $m \geq N$. Then it is possible to select a sequence $(y_n)_{n\in\mathbb{N}}$ in A_N with $d(x_n, y_n) < 2\epsilon$. In fact,

$$A \subseteq (A_N)_{2\epsilon}.$$

Since A_N is compact, take $y \in A_n$, such that $B_\epsilon(y)$ contains infinitely many y_n. Then $B_{3\epsilon}(y)$ contains infinitely many x_n. The same argument holds for each subsequence of $(x_n)_{n\in\mathbb{N}}$. Thus for all $\epsilon > 0$ in each subsequence of $(x_n)_{n\in\mathbb{N}}$ there is another subsequence $(x_{n_k})_{k\in\mathbb{N}}$ with

$$d(x_{n_k}, x_{n_l}) < \epsilon \qquad \text{for all } k, l \in \mathbb{N}$$

Using induction, we can now construct a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ such that

$$d(x_{n_k}, x_{n_{k+1}}) \leq \frac{1}{2^k}$$

Since X is complete, x_n contains convergent subsequence. So A is compact.

It remains to show that $A_n \to A$. Using the same arguments as in the previous paragraph, we can find for each $\epsilon > 0$ an $N \in \mathbb{N}$ with

$$A \subseteq (A_n)_\epsilon \qquad \text{for all } n \geq N.$$

For any $x_n \in A_n$, we construct a sequence such that

$$d(x_m, x_{m+1}) \leq \frac{1}{2^m} \qquad \text{for all } m \geq n.$$

This sequence converges to $x \in A$ with

$$d(x_n, x) < \frac{1}{2^n}.$$

Thus, if $1/2^n < \epsilon$,

$$A_n \subseteq (A)_\epsilon.$$

This finishes the proof. □

6.6 Locally Compact Spaces

6.24. Definition: A space X is called locally compact, if each point has a compact neighborhood.

6.25. Problem: Proof that, if X is locally compact and Hausdorff, each neighborhood contains a compact neighborhood.

6.26 Theorem: **(Alexandrov)** *Let X be locally compact and Hausdorff. If $\infty \notin X$, we can define a topology on*

$$\tilde{X} = X \cup \{\infty\}$$

such that $\tilde{X}$ is compact and Hausdorff, and X has the relative topology on $\tilde{X}$.

Proof: Set the open sets on $\tilde{X}$ as the open sets in X, or the complements of compact subsets of X. Since compact sets are closed, their complements are open in X, and so X has the relative topology in $\tilde{X}$.

It remains to show that $\tilde{X}$ is compact and Hausdorff. But any open covering covers ∞ with one set, and the compact complement with the rest of the sets. The proof that $\tilde{X}$ is Hausdorff remains as an exercise. $\square$

Chapter 7

Filters

7.1. Definition: $\mathcal{F} \subseteq \mathcal{P}(X)$ is called **filter** on X, if

$$A, B \in \mathcal{F} \;\Rightarrow\; A \cap B \in \mathcal{F},$$
$$A \in \mathcal{F}, A \subseteq B \;\Rightarrow\; B \in \mathcal{F},$$
$$\emptyset \notin \mathcal{F}, \quad X \in \mathcal{F}.$$

7.2. Definition: $\mathcal{B} \subseteq \mathcal{F}$ is a **basis** of the filter $\mathcal{F}$, if for all $F \in \mathcal{F}$ exists a $B \in \mathcal{B}$ with $B \subseteq F$. Thus

$$\mathcal{F} = \dot{\mathcal{B}} := \{F : B \subseteq F \text{ for some } B \in \mathcal{B}\}.$$

If this is a filter, $\dot{\mathcal{B}}$ is called the filter **generated by** $\mathcal{B}$. A set $\mathcal{B}$ is called a **filter basis**, if $\dot{\mathcal{B}}$ is indeed a filter.

7.1 Examples

7.3. Example: The neighborhood filter $\mathcal{U}(x)$ in a topology. In a metric space, the countable set

$$\{B_{1/n} : n \in \mathbb{N}\}$$

is a basis of the neighborhood filter.

7.4. Example: $\dot{A}$ the filter of the supersets of A, and $\dot{x}$ the filter of the sets containing x.

7.5. Problem: Proof that $\mathcal{B} \subseteq \mathcal{P}(X)$ is a filter basis, if and only if each finite intersection of sets in $\mathcal{B}$ contains an element of $\mathcal{B}$ and $\mathcal{B}$ is not empty and does not contain $\emptyset$.

7.6. Problem: Find necessary and sufficient conditions for a set $\mathcal{B} \subseteq \mathcal{P}(X)$ to be contained in a filter. What is the smallest filter containing this set?

7.2 Convergence

7.7. Definition: A filter $\mathcal{F}$ on a topology X converges to x ($\mathcal{F} \to x$), if

$$\mathcal{U}(x) \subseteq \mathcal{F}$$

($\mathcal{F}$ is **finer** than $\mathcal{U}(x)$).

7.8. Example: $\dot{x} \to x$ in any topology.

7.9. Problem: Let $f : X \to Y$ and $\mathcal{F}$ a filter on X. Then

$$\{f(F) : F \in \mathcal{F}\}$$

is a filter basis on Y. If f is surjective, it is a filter. We denote the filter generated by this set by $f(\mathcal{F})$.

7.10 Theorem: *$f : X \to Y$ is continuous, if and only if*

$$\mathcal{F} \to x \;\Rightarrow\; f(\mathcal{F}) \to f(x)$$

for all filters $\mathcal{F}$.

7.11. Problem: Use this to show that the concetanation of continuous mappings is continuous.

7.3 Metric Spaces

7.12. Problem: For a sequence $(x_n)_n$ in X define the filter $\mathcal{F}((x_n)_n)$ generated by the filter basis of end pieces

$$\{x_n : n \geq k\}$$

for $k \in \mathbb{N}$. Proof

$$x_n \to x \;\Leftrightarrow\; \mathcal{F}((x_n)_n) \to x.$$

in metric spaces.

7.13. Problem: Show that the filter generated by a subsequence is finer than the filter generated by the sequence. Find a generalization of accumulation points in metric spaces in terms of finer filters. Find an equivalent statement in terms of neighborhood of the accumulation point.

Chapter 8

Ultrafilters

8.1. Definition: $\mathcal{F}$ is called an **ultrafilter** on X, if there is no proper finer filter.

8.2 Theorem: *$\mathcal{G}$ is an ultrafilter on X, if and only if*

$$A \in \mathcal{G} \text{ or } A^c \in \mathcal{G}$$

($A^c = X \setminus A$) for all $A \subseteq X$.

8.3. Example: The only ultrafilter we can write down is $\dot{x}$.

8.1 Zorn's Lemma

8.4 Theorem: *Every filter is contained in a finer ultrafilter.*

Proof: The proof is based on Zorn's lemma: If in a partially ordered set all ordered subsets have an upper limit, then the set has a maximal element. We apply this to the set of all finer filters. Of course, this is all based on the axiom of choice. □

8.2 Compact Spaces

8.5 Theorem: *X is compact, if and only if each ultrafilter in X converges.*

Proof: "'⇒"': If the ultrafilter $\mathcal{G}$ does not converge, for each $x \in X$ there is and open $U \in \mathcal{U}(x)$, $U \notin \mathcal{G}$. This covering cannot contain a finite subcovering $U_1, \ldots, U_n$, since otherwise

$$\emptyset = (U_1 \cup \ldots \cup U_n)^c = U_1^c \cap \ldots \cap U_n^c \in \mathcal{G}.$$

"'$\Leftarrow$"': Assume an open covering $\mathcal{M}$ without finite covering. Then the set of complements of finite unions in $\mathcal{M}$ is a filter basis, since it does not contain the empty set. A finer ultrafilter $\mathcal{G}$ cannot converge. Otherwise, take the limit x and $x \in U \in \mathcal{M}$. Since $\mathcal{G} \to x$,

$$(U_1 \cup \ldots \cup U_n)^c \subseteq U$$

for some $U_1, \ldots, U_n \in \mathcal{M}$. This is equivalent to

$$X = U \cup U_1 \cup \ldots \cup U_n$$

and $\mathcal{M}$ would have a finite covering. This part is basis on the axiom of choice. □

8.6. Remark: Thus each filter in a compact space has an accumulation point.

8.7. Problem: Take the cofinite topology on $\mathbb{N}$, and the filter generated by $(n)_{n\in\mathbb{N}}$. What are the accumulation points of this filter?

8.8. Problem: The intersection of filters is obviously a filter. Show that an ultrafilter, which is finer than an intersection of ultrafilters, is equal to one of them.

8.9. Problem: If $\mathcal{G}$ is an ultrafilter on X, and $f : X \to Y$ is surjective, then

$$\{f(F) : F \in \mathcal{G}\}$$

is an ultrafilter on Y.

8.3 Product of Compact Spaces

8.10 Theorem: **(Tihonov)** *The product of compact topologies is compact.*

Proof: The projections of an ultrafilter $\mathcal{G}$ on $\prod_{i\in I} X_i$ to each X_i is an ultrafilter, converging to some x_i for all $i \in I$. It remains to prove, that $\mathcal{G}$ converges to $x = (x_i)_{i\in I}$. However, each neighborhood of x contains an open set of the form

$$U = \pi_{i_1}^{-1}(U_{i_1}) \cap \ldots \cap \pi_{i_n}^{-1}(U_{i_n})$$

for some indices $i_1, \ldots, i_n$, where U_{i_k} are neighborhoods of x_{i_k}, and therefore in $\pi_{i_k}(\mathcal{G})$, so $\pi_{i_k}(G_k) = U_{i_k}$ for some $G_k \in \mathcal{G}$. Since

$$G_k \subseteq \pi_{i_k}^{-1}(\pi_{i_k}(G_K)) = \pi_{i_k}^{-1}(U_{i_k})$$

it follows that $\pi_{i_k}^{-1}(U_{i_k})$ is in $\mathcal{G}$ for all k. Thus $U \in \mathcal{G}$. □

8.11. Problem: Proof this theorem for finite products with the help of open coverings, and without using the axiom of choice.

8.12. Problem: Let $I = [0, 1]$ and $X_i = [0, 1]$ for each $i \in I$. By Tihonov's theorem the product space $X = \prod_i X_i$ is compact (we write $X = I^I$). Proof that X is not sequence-compact. To do this, define $f_n(x)$ as the n-th digit in the decimal expansion of x (divided by 10), and show that it has not convergent subsequence. Here, a sequence converges in a topology to x, if each neighborhood of x contains an end piece of the sequence.

8.4 Stone-Cech Compactification

8.13 Theorem: *Let X be a completely regular topological space, and $C^*(X)$ the set of all continuous mappings from E to $I = [0,1]$. Set*

$$\tilde{X} = \prod_{f \in C^*(X)} I_f$$

where $I_f = I$ for all f. Then we can define the mapping $e : X \to \tilde{X}$ by

$$e(x) = (f(x))_{f \in C^*(X)}.$$

*I.e., $e(x)$ evaluates all functions at x. Then e is a homeomorphism between X and $e(X)$. This compactification is called the **Stone-Ĉech Compactification**.*

Proof: Since each component mapping $\pi_f(e(x)) = f(x)$ of e is continuous, e is continuous. It is immediately clear, that e is injective.

Now, let U be an open neighborhood of $x \in X$. Then there is a continuous function $f_0 : X \to [0,1]$ such that

$$f_0(x) = 1, \quad f_0(X \setminus U) = 0.$$

The set

$$V = \pi_{f_0}^{-1}]1/2, \infty[$$

is open in $\tilde{X}$. We show $e^{-1}(V) \subseteq U$. Now

$$t \in e^{-1}(V) \;\Leftrightarrow\; e(t) \in V \;\Leftrightarrow\; \pi_{f_0}(e(t)) = f_0(t) > 1/2$$

The latter implies $t \in U$. $\square$

8.14. Remark: The embedding e is called a **Tihonov embedding**.

Chapter 9

Nets

9.1. Definition: An index set I is called directed by "'$\leq$"', if

$$i \leq i$$
$$i \leq j, j \leq k \;\Rightarrow\; i \leq k$$

for all $i, j, k \in I$, and if for all $i, j \in I$ there is a $k \in I$ with $i \leq k$ and $j \leq k$.

9.2. Problem: Show that $\mathcal{U}(x)$ is directed by the superset relation.

9.3. Problem: Show that the set of all finite subsets of a given set is directed with the subset relation.

9.4. Definition: A **net** is a mapping $x : I \to X$ from a directed set x to a set X, usually written as $(x_i)_{i \in I}$.

9.5. Remark: A sequence is a special net with $I = \mathbb{N}$ and the usual ordering.

9.1 Convergence

9.6. Definition: A net $(x_i)_{i \in I}$ in a topological space X converges against $x \in X$, if all neighborhoods $U \in \mathcal{U}(x)$ contain an end piece

$$N_k = \{x_i : k \leq i\}$$

of the net.

9.7. Problem: Assure that this agrees with convergence of sequences in metric spaces. Show that in a Hausdorff space, each net can have only one limit.

9.8. Problem: Proof that the closure of a set $A \subseteq X$ consists of all points x, such that there exists a net in A converging to x. Use the net

$$(x_U)_{U \in \mathcal{U}(x)}, \quad x_U \in U$$

in the proof with the superset order on $\mathcal{U}(x)$. Note that you have to use the axiom of choice to define this net. This means, that all topological spaces are completely defined by their converging nets.

9.9. Problem: Proof that a mapping $f : X \to Y$ is continuous in x, if and only if

$$x_i \to x \;\Rightarrow\; f(x_i) \to f(x)$$

for each net $(x_i)_{i \in I}$.

9.10. Problem: Let X_j, $j \in J$, be topological spaces, and $f_j : X \to X_j$ be mappings, X equipped with the induced topology. Show that a net $(x_i)_{i \in I}$ in X converges to $x \in X$, if and only if the net $(f_j(x_i))_{i \in I}$ converges to $f_j(x) \in X_j$ for all $j \in J$.

9.2 Compact Sets

9.11. Definition: A point x in a topological space X is called an accumulation point of the net $(x_i)_{i \in I}$, if each neighborhood contains at least one point in each end piece of the net.

9.12 Theorem: *A space X is compact, if and only if each net has an accumulation point.*

Proof: "'$\Rightarrow$"' Assume there is a net $(x_i)_{i \in I}$ without an accumulation point. Then each point $y \in U$ has an open neighborhood U_y, such that one endpiece N_{i_y} is completely contained in U_x^c. There is a finite covering $U_{y_1}, \ldots, U_{y_n}$ of X. Take an index $i \in I$ with

$$i_{x_1} \leq i, \ldots, i_{x_n} \leq i.$$

Then $y_i \in X$ cannot be in any U_{x_k}, $k = 1, \ldots, n$. This is a contradiction.

"'$\Leftarrow$"': Assume an open covering $\mathcal{M}$ has no finite covering. We define a filter on X in the following way. The index set I consists of all finite subsets of $\mathcal{M}$ with the subset relation. For $i = \{M_1, \ldots, M_n\} \in I$ we choose

$$x_i \notin M_1 \cup \ldots \cup M_n,$$

using the axiom of choice. Then $(x_i)_{i \in I}$ cannot have an accumulation point x, since $x \in M$ for some $M \in \mathcal{M}$, and the endpiece starting with $i = \{M\}$ has no element in M. □

9.3 Subnets

9.13. Definition: A subnet of a net $(x_i)_{i \in I}$ is a net $(x_{\phi(j)})_{j \in J}$ where $\phi : J \to I$ has the property, that for each $i_0 \in I$ there is a $j_0 \in J$ such that

$$\phi(j) \geq i_0$$

for all $j \geq j_0$.

9.14. Problem: Proof that each subnet converges to x if the net converges to x.

9.15. Problem: Proof that a point x is an accumulation of a net, if and only if there is a subnet converging to x. In the proof, use the subnet

$$J = \{(U, i) : U \in \mathcal{U}(x),\, i \in I\}$$

with a proper relation "'$\leq$"' and $x_{\phi(j)} \in U$ for $j = (U, i)$ chosen with $\phi(j) \geq i$.

9.16. Remark: The subnet may have a larger index set than the original set. E.g., we have seen that on $[0, 1]^{[0,1]}$ there are sequences without a convergent subsequence. But each sequence contains a convergent subnet, since this space is compact.

Chapter 10

Connected Spaces

10.1. Definition: A topological space is **connected** if it is not the union two open, non-empty, disjoint sets.

10.2. Problem: Formulate that in more detail for subsets of a topological space with the relative topology.

10.3. Problem: Proof that the connected subsets of $\mathbb{R}$ are the intervals.

10.1 Images of Connected Sets

10.4 Theorem: *The continuous image of a connected space is connected.*

10.5. Problem: Proof the theorem.

10.2 Path Connected Sets

10.6. Definition: A topological space is **path connected**, if any two points can be connected by a continuous path.

10.7. Example: A special example are convex and star like sets in any real vector space.

10.8. Problem: Proof that any path connected space is connected.

10.9. Problem: Proof that there is no continuous and injective mapping from an open, non-empty subset of $\mathbb{R}^n$, $n \geq 2$, to $\mathbb{R}$. Use the fact, that the punctured disk $B_r(x) \setminus \{x\}$ is connected.

10.10. Problem: In a normed vector space, proof that any connected open set is path connected. To prove this, show that the set of points, that can be reached from one given point, is open. Therefore it must be the complete set.

10.11. Example: The graph G of the function $f :]0, \infty[\to \mathbb{R}$

$$f(x) = \sin\left(\frac{1}{x}\right)$$

together with the line segment

$$S = \{(0, y) : y \in [-1, 1]\}$$

is connected, but not path connected. To prove that it is connected, assume two open, non-empty sets covering $I \cup G$. One of them must contain I, and by a compactness argument, we get that G is not connected. However, G is obviously path connected. That $I \cup G$ is not path connected is left as a problem.

10.3 Union of Connected Sets

10.12 Theorem: *The union of connected subsets of a topological space X is connected, if each two of the sets have a point in common.*

10.13. Problem: Proof the theorem.

10.4 Connected Components

10.14. Definition: The largest connected set containing a point x in a topological space is called the **connected component** $Z(x)$ of x.

10.15. Problem: Proof that the connected component exists and is unique, and that two connected components are either equal or disjoint.

10.16. Problem: Proof that the connected component of a point is closed. Prove that it is also open, if the space has only finitely many connected components.

10.17. Problem: Proof that the connected components are open in a space, if each point has a connected neighborhood.

10.18. Problem: In a space we can define the set of all points $\tilde{Z}(x)$, which can be connected with a path to x. Proof that this set is open in a normed vector space. Conclude, that an open and connected set in a normed vector space is path connected.

10.5 Product Spaces

10.19. Problem: Proof that the product of two connected spaces is connected. Use the fact, that the mapping $x \mapsto (x, y)$ from X to $X \times Y$ is an embedding, and conclude, that any two points are in the same connected component.

10.20 Theorem: *The product $X = \prod_i X_i$ of non-empty spaces X_i, $i \in I$ is connected, if and only if all X_i, $i \in I$, are connected.*

Proof: "'$\Rightarrow$"': Left as a trivial problem.

"'$\Leftarrow$"': In the problem, we proved this for two spaces only, and thus for finitely many spaces. Assume X is covered by disjoint, non-empty open sets U and V, and fix $u \in U$ and $v \in V$. Then there are $i_1, \ldots, i_n$ in I such that

$$x \in \bigcap_k \pi_{i_k}^{-1}(U_{i_k}) \subseteq U, \quad y \in \bigcap_k \pi_{i_k}^{-1}(V_{i_k}) \subseteq V,$$

for open sets U_{i_k} and V_{i_k}, $k = 1, \ldots, n$. The space

$$E = \prod_k X_{i_k}$$

can then be embedded into X by setting

$$\pi((x_k)_{1 \leq k \leq n}) = (t_i)_{i \in I}$$

with

$$t_i = \begin{cases} x_k, & i = i_k \text{ for some } k, \\ p_i, & i \neq i_k \text{ for all } k, \end{cases}$$

for fixed points $p_i \in X_i$. Thus $F = \pi(E)$ is connected. But F is covered by the disjoint, non-empty open sets $F \cap U$ and $F \cap V$ (details left as a problem). □

10.21. Problem: Let X and Y be connected topological spaces, A be a proper subset of X, and B be a proper subset of Y. Proof that

$$(X \times Y) \setminus (A \times B)$$

is connected.

Chapter 11

Separable Spaces

11.1. Definition: A set M is **dense** in a topological space X, if $\overline{M} = X$. A topological space with a countable dense subset is called **separable**.

11.2. Example: $\mathbb{R}^n$, and thus each finite dimensional normed space, is separable.

11.3. Example: A discrete space is separable, only if it is countable.

11.1 First and Second Axiom of Countability

11.4. Definition: A topological space is **first-countable**, if each point has a countable neighborhood basis. It is called **second-countable**, if its topology has a countable base.

11.5. Example: Metric spaces are first-countable.

11.6. Example: $\mathbb{R}^n$, and thus each finite dimensional normed space, is second-countable.

11.7. Problem: Show that any second-countable space is first-countable.

11.8. Problem: Show that every second-countable topological space is separable.

11.9. Problem: Show that every separable first-countable space is second-countable.

11.10. Problem: Proof that every regular second-countable space is normal. To do this, for two disjoint closed sets A and B, construct sequences U_n and V_n of open sets such that

$$A \subseteq \bigcup U_n \subseteq \bigcup \overline{U_n} \subset X \setminus B, \quad B \subseteq \bigcup V_n \subseteq \bigcup \overline{V_n} \subset X \setminus A.$$

Then set

$$V'_n = V_n \setminus \bigcup_{k \le n} \overline{W_k}, \quad W'_n = W_n \setminus \bigcup_{k \le n} \overline{V_k}.$$

Finally, show that

$$V = \bigcup V'_n, \quad W = \bigcup W'_n$$

separates A from B.

11.11 Theorem: *Let X be a second-countable topological T1 space. Then the following are equivalent.*

(1) *X is regular.*

(2) *X is normal.*

(3) *There is a metric on X generating the topology.*

(4) *X is homeomorphic to a subset of $[0,1]^I$, where I is a countable index set.*

Proof: By the above problem, (1) is equivalent to (2). We already know that (3) implies (2), and since the product space in (4) is generated by a metric, the toplogoy of each space homeomorphic to a subspace is also generated by a metric. So (4) implies (3).

It remains to show that (2) implies (4). By the Urysohn Lemma, we know that X is completely regular. Let $\mathcal{B}$ be a countable basis of X. Then

$$I = \{(U,V) : U,V \in \mathcal{B}, \overline{U} \subseteq V\}$$

is countable. For each $i = (U,V) \in I$ there is a function $f_i : X \to [0,1]$ with

$$f_i(\overline{U}) = 1, \quad f_i(X \subseteq V) = 0.$$

We now embed X into $[0,1]^I$ by the Tihonov embedding

$$e(x) = (f_i(x))_{i \in I}.$$

The proof, that this is an embedding is similar as in the Tihinov compactness theorem and left as a problem. □

Chapter 12

Spaces of Mappings

12.1. Definition: For a set X and a metric space Y, we define a metric

$$d(f,g) = \max\left\{1, \sup_{x\in X} d(f(x), g(x))\right\}$$

on the set $\mathcal{A}(X,Y)$ of all mappings from X to Y. It is called the metric of **uniform convergence**.

12.2. Problem: Show that d is indeed a metric.

12.3. Problem: The set $\mathcal{A}(X,Y)$ is identical to the product Y^X. Show that topology of uniform convergence is finer than the product topology on Y^X, which is called the topology of **pointwise convergence**.

12.4. Problem: Show that $\mathcal{A}(X,Y)$ is complete, if and only if Y is complete.

12.1 The Space of Continuous Mappings

12.5. Definition: For a topological spaces X and Y, we define $C(X,Y)$ as the set of all continuous mappings from X to Y.

12.6. Problem: Let X be a compact Hausdorff space, and Y be a metric space. Then

$$\tilde{d}(f,g) = \sup_{x\in X} d(f(x), g(x))$$

is a metric on $C(X,Y)$. It generates the relative topology on $C(X,Y)$ in the topological space $\mathcal{A}(X,Y)$.

12.7. Problem: For a topology X and a metric space Y, show that $C(X,Y)$ is closed in $\mathcal{A}(X,Y)$ with the topology of uniform convergence. This is equivalent to the following theorem.

12.8 Theorem: *Let X be a topology, and Y be a metric space. The limit of a uniformly convergent sequence of continuous functions in $\mathcal{A}(X,Y)$ is continuous.*

12.9 Theorem: **(Dini)** *Let X be compact, and $f_n : X \to \mathbb{R}$ be a sequence of continuous functions, which are pointwise monotone increasing. If f_n converges pointwise to a continuous function $f : X \to \mathbb{R}$, then it converges uniformly.*

12.10. Problem: Proof this theorem.

12.11. Example: Proof inductively that the sequence of polynomials

$$p_0(x) = x, \quad p_{n+1}(x) = p_n(x) + \frac{1}{2}(x - p_n(x)^2)$$

is uniformly increasing on [0,1], and bounded by $\sqrt{x}$. Use the fact that $f(t) = t - t^2/2$ is monotone increasing in $[0, 1]$. Conclude, that $p_n(x)$ converges pointwise to $\sqrt{x}$, and thus uniformly.

12.12. Problem: Show that the limit function must be assumed to be continuous for the Dini theorem to hold.

12.2 Uniform Continuity

12.13. Definition: For two metric spaces X and Y, a function $f \in C(X, Y)$ is called **uniformly continuous**, if for all $\epsilon > 0$ exists a $\delta_\epsilon > 0$, such that

$$d(x, y) < \delta_\epsilon \;\Rightarrow\; d(f(x), f(y)) < \epsilon$$

for all $x, y \in X$. A set of functions is called **equi-continuous**, if the same δ_ϵ can be found for all functions f in the set.

12.14. Problem: Show that $f : \,]0, 1[\, \to \mathbb{R}$ defined by $f(x) = 1/x$ is not uniformly continuous. Make examples of bounded continuous functions $f : \mathbb{R} \to \mathbb{R}$, which are not uniformly continuous.

12.15 Theorem: *Let X be a compact metric space, and Y be a metric space. Then all $f \in C(X, Y)$ are uniformly continuous.*

Proof: Let $\epsilon > 0$. By continuity, for each $x \in X$, there is a $\delta_x > 0$ such that

$$f(B_{\delta_x}(x)) \subseteq B_{\epsilon/2}(f(x)).$$

We can cover X with finitely many $B_{\delta_x/2}(x)$. Let δ be the smallest of these δ_x. Then

$$d(s, t) < \frac{\delta}{2} \;\Rightarrow\; d(f(s), f(t)) < \epsilon$$

for all $s, t \in X$ (details as problem). □

12.16. Problem: Find a proof using sequence compactness.

12.17. Problem: On the set of all linear functions Φ from a normed vector space X to a normed vector space Y, we can define

$$\|\Phi\| = \sup\{\|\Phi(x)\| : x \in X \text{ with } \|x\| = 1\}.$$

From this, we get

$$\|\Phi(x)\| \leq \|\Phi\| \, \|x\| \qquad \text{for all } x \in X.$$

Show that Φ is continuous, if and only if $\|\Phi\| < \infty$, and show that it is uniformly continuous in this case. Show that $\|\Phi\|$ is a norm on the vector space of all continuous, linear functions $\mathcal{L}(X, Y)$ from X to Y. If we restrict all Φ to a bounded set B, then convergence in this norm is equivalent to uniform convergence on B. The dual ball

$$\mathcal{B}_1 = \{\Phi : \|\Phi\| \leq 1\}$$

is equi-continuous.

12.3 The Arzelà-Ascoli Theorem

12.18. Problem: Let X be a compact metric space, Y be a metric space, and $(f_n)_{n\in\mathbb{N}}$ be an equi-continuous sequence in $C(X, Y)$. Show that if $f_n(x)$ converges pointwise on a dense subset of X, it converges everywhere, the limit function is continuous, and the convergence is uniform.

12.19 Theorem: **(Arzelà-Ascoli)** *Let X be a separable and compact metric space, and Y be a compact metric space. Then each equi-continuous sequence $(f_n)_{n\in\mathbb{N}}$ has a uniformly convergent subsequence.*

Proof: We restrict the functions on the countable dense subset M of X. Then Y^M is compact, and since M is countable, the product topology is generated by a metric. So the sequence $((f_n(x))_{x\in M})_{n\in\mathbb{N}}$ has a convergent subsequence in Y^M. Thus $(f_n)_{n\in\mathbb{N}}$ has a subsequence, which converges pointwise on M. The result follows from the problem above. □

12.20. Problem: Extend this result in the following way: If X is separable, and

$$M_x = \{f_n(x) : n \in \mathbb{N}\}$$

is contained in a compact subset of Y for all x, then each equi-constinous sequence contains a subsequence, which converges uniformly on all compact subsets of X.

12.21. Example: Let $f : [a, b] \times \mathbb{R} \to \mathbb{R}$ be continuous. We want to solve the initial value problem

$$y'(x) = f(x, y(x)), \quad y(a) = c.$$

With the Euler method, we can find a sequence of piecewise linear functions $y_n : [a, b] \to \mathbb{R}$ with $y_n(a) = c$ and

$$y_n(x_{n,k+1}) = y_n(x_{n,k}) + f(x_{n,k}, y_n(x_{n,k})) \cdot (x_{n,k+1} - x_{n,k})$$

for a partition

$$a = x_{n,0} < x_{n,1} < \ldots < x_{n,n} = b.$$

If y_n converges uniformly, the limit function f for a sequence of partitions with maximal interval size tending to 0 is continuous and thus Riemann-integrable, and it follows that

$$y(x) - y(a) = \int_a^x f(t, y(t))\, dt$$

for all $x \in [a, b]$. Thus y is a solution to our initial value problem. In fact, it is enough to get a uniformly converging subsequence. So we need to assure that the sequence y_n is equi-continuous and bounded. However, in a neighborhood of (a, c) we may assume that f is bounded by a constant C. Hence, if $b - a$ is small enough, the sequence y_n will be bounded. Essentially, we have proved the following theorem.

12.22 Theorem: *Let $f : [a, b] \times U \to \mathbb{R}^n$ be continuous, $U \subseteq \mathbb{R}^n$ open. Then for all $c \in U$ the initial value problem*

$$y'(x) = f(x, y(x)), \quad y(a) = c.$$

has a solution in some interval $[a, a + \epsilon]$, $\epsilon > 0$.

12.23. Problem: Work out the missing details of this generalization.

12.24. Problem: Provide an example that the solution is not unique. Show that it must be unique, if $f(x, y)$ is Lipschitz continuous with respect to y. In this case, proof the existence with the Banach fixed point theorem.

12.25 Theorem: *Let X be a separable Banach space, $(F_n)_{n\in\mathbb{N}}$ be a sequence of continuous linear functionals $F_n : X \to Y$ from X to some normed vector space Y such that*

$$\|F_n\| \leq M < \infty \qquad \text{for all } n \in \mathbb{N}.$$

Then there exists a subsequence converging pointwise to a continous linear functional F with $\|F\| \leq M$.

Proof: The sequence has a subsequence, converging on all compact subsets of X to a linear functional (details left as problem). Since points are compact, the theorem follows. □

12.4 The Compact-Open Topology

12.26. Definition: With the set $\mathcal{K}$ of all compact subsets of Hausdorff space X, and a metric space Y, we can define a mapping

$$\phi(f) = \prod_{K \in \mathcal{K}} f|_K$$

for all which maps $f \in C(X, Y)$ into the product space

$$\prod_{K \in \mathcal{K}} C(K, Y).$$

We assume that all $C(K, Y)$ are equipped with the metric of uniform convergence. Then the topology on $C(X, Y)$ induced by ϕ is called the **compact-open** topology or the topolgy of **compact convergence**.

12.27. Problem: The same definition can be generalized to system of other sets $\mathcal{K} \subseteq \mathcal{P}(X)$. Proof that the induced topology is the the topology of pointwise convergence, if $\mathcal{K}$ is the set of all singletons $\{x\}$, $x \in X$, and that it is the topology of uniform convergence, if $\mathcal{K} = \{X\}$.

12.28. Problem: Proof that the compact-open topology is Hausdorff, and that it is regular, if X has this property.

12.29. Problem: Proof that the open-compact topolgy on $C(X, Y)$ is generated by a metric, if X is **sigma-compact**, i.e., there exists a sequence of compact subsets, such that the union of the sequence is X.

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