Book of abstracts

11th Workshop of the GAMM Activity Group "Analysis of Partial Differential Equations"

Invited talks

Manuel Friedrich (FAU Erlangen-Nürnberg)

Nonlinear and linearized models in thermoviscoelasticity

In this talk, I present a quasistatic nonlinear model in thermoviscoelasticity at a finite-strain setting in the Kelvin-Voigt rheology where both the elastic and viscous stress tensors comply with the principle of frame indifference under rotations. The force balance is formulated in the reference configuration by resorting to the concept of nonsimple materials whereas the heat transfer equation is governed by the Fourier law in the deformed configurations. Weak solutions are obtained by means of a staggered in-time discretization where the deformation and the temperature are updated alternatingly. Our result refines a recent work by Mielke and Roubíček since our approximation does not require any regularization of the viscosity term. Afterwards, we focus on the case of deformations near the identity and small temperatures, and we show by a rigorous linearization procedure that weak solutions of the nonlinear system converge in a suitable sense to solutions of a system in linearized thermoviscoelasticity. Based on joint work with Rufat Badal (Erlangen) and Martin Kružík (Prague).

Yves van Gennip (TU Delft)

Discrete-to-continuum limits of graph-based gradient flows

Gradient flows on graphs are common ingredients in machine learning methods. Discrete-to-continuum limits of such flows are of interest since they establish consistency of the method. In this talk we will have a look at the limits of two such flows formulated in terms of a variational inequality: the total variation flow and the one-dimensional Allen–Cahn flow. If time allows, we can also take a look at semi-group methods to establish such limits.

Franz Gmeineder (Universität Konstanz)

Quasiconvexity, (p, q)-growth and partial regularity

We display new results on the regularity properties of relaxed minimizers of quasiconvex functionals of (p, q)-growth. These results apply to the natural range for which the functionals can be meaningfully extended (or relaxed) and apply to signed integrands as well. This extends previously known exponent ranges of Schmidt in a basically optimal way. Moreover, despite being natural in view of coercivity, signed, quasiconvex allow for different phenomena that are invisible in the convex situation. Specifically, some focus will be put on the non-availability of measure representations à la Fonseca & Malý for the relaxed functionals and, more importantly, why they are not really required for partial regularity. Based on joint work with Jan Kristensen.

Moritz Kaßmann (Universität Bielefeld)

Variational methods for nonlocal problems

We review recent results on nonlocal problems with a focus on variational methods. First, we explain trace and extension results for Sobolev-type function spaces that are well suited for nonlocal Dirichlet and Neumann problems including those for the fractional p-Laplacian. The main focus is on optimal choice of exterior data. The results are robust with respect to the order of differentiability. In this sense they are in align with the classical trace and extension theorems. This part of the talk is based on a recent joint work with Floran Grube, see arXiv:2305.05735. Second, we discuss the local regularity program for weak solutions to linear parabolic nonlocal equations with bounded measurable coefficients. Within the variational framework we establish the parabolic Harnack inequality and Hölder regularity estimates. This part of the talk is based on a recent joint work with Marvin Weidner, see arxiv:2303.05975.

Nikos Katzourakis (University of Reading)

Existence, uniqueness and characterisation of local minimisers in higher order Calculus of Variations in L^∞

The study of higher order supremal functionals is very novel in the Calculus of Variations in L^∞, and a systematic theory is at present underway. They exhibit a strikingly different behaviour compared to first order functionals, for which there exists an established theory, pioneered by Aronsson in 1960s. In this talk I will discuss some new results, in which we have proved that, under appropriate conditions, "localised" minimisers of second order functionals can be characterised as solutions to a nonlinear system of PDEs. This is different in nature from the corresponding Aronsson equation. As we have shown, surprisingly, the Aronsson equation is only a necessary, but not a sufficient condition for minimality. We have also established the existence of localised minimisers, characterising them through L^p approximations, via a method of penalisation. Further, we have shown that localised minimisers are unique, even when lower order terms are allowed. Based on joint work with Roger Moser (Bath, UK).

Christina Lienstromberg (Universität Stuttgart)

Long-time behaviour of solutions to non-Newtonian thin-film equations

I will offer an insight into mathematical models describing the dynamic behaviour of non-Newtonian thin-films flows. The resulting PDEs are in general nonlinear, (doubly) degenerate, of fourth order, and with a possibly 'weak' dependence of the coefficients on the unknown.

We discuss recent results on the long-time behaviour of solutions to an evolution equation for the surface of a non-Newtonian fluid film with power-law rheology in the Taylor–Couette setting. In the case of shear-thickening fluids, one observes that solutions which are initially close to a steady state, converge to equilibrium in finite time. In the shear-thinning case, we find that steady states are polynomially stable in the sense that, as time tends to infinity, solutions which are initially close to a steady state, converge to equilibrium at rate $1/t^{1/\beta}$ for some $\beta > 0$.

The talk is based on joint work with Tania Pernas-Castaño, Juan Velázquez (both Bonn).

Malte Peter (Universität Augsburg)

Identification of microstructural information from macroscopic boundary measurements in linear elasticity

We consider the upscaled linear elasticity problem in the context of periodic homogenization. Based on measurements of the deformation of the (macroscopic) boundary of a body for a given forcing, it is the aim to deduce information on the geometry of the microstructure. For a parametrized microstructure, we are able to prove that there exists at least one solution of the associated minimization problem based on the L^2-difference of the measured deformation and the resulting deformation for a given parameter. To facilitate the use of gradient-based algorithms, we derive the Gâteaux derivatives using the Lagrangian method of Céa, and we present numerical experiments showcasing the functioning of the method.

This is joint work with T. Lochner (University of Augsburg).

Stefanie Sonner (Radboud Universiteit)

Degenerate reaction diffusion systems arising in the modelling of biofilms: well-posedness, regularity and travelling waves solutions

Biofilms are dense aggregations of bacterial cells attached to a surface and held together by a self-produced matrix of extracellular polymeric substances. They affect many aspects of human life and play a crucial role in natural, medical and industrial settings. We consider continuum models for spatially heterogeneous biofilm communities formulated as quasilinear reaction diffusion systems. Their characteristic and challenging feature is the two-fold degenerate diffusion coefficient for the biomass density comprising a polynomial degeneracy (as known from the porous medium equation) and a fast diffusion singularity as the biomass density approaches its maximum value. This degenerate equation is coupled to a semilinear reaction diffusion equation or an ordinary differential equation for the nutrient concentration. The latter case models cellulolytic biofilms where nutrients are immobilized in a cellulose surface that supports the biofilm and that is degraded by the bacteria. In this talk we present results on the well-posedness and regularity of solutions for such systems on bounded domains with Dirichlet and mixed Dirichlet-Neumann boundary conditions as well as for the Cauchy problem. For systems with immobilized nutrients, i.e. if the degenerate equation is coupled to an ordinary differential equation, we also prove the existence of traveling wave solutions. Invading fronts had earlier been observed in biological experiments on cellulolytic biofilms as well as in numerical simulations of the model.

Marita Thomas (FU Berlin)

Approximating dynamic phase-field fracture with a first-order formulation for velocity and stress

We investigate a model for dynamic fracture at small strains. The sharp crack interface is regularized with a phase-field approximation. For the phase-field variable a viscous evolution with a quadratic dissipation potential is employed and a non-smooth penalization prevents material healing. For the solid material both the case of a visco-elastic and of a purely elastic constitutive law is considered. The momentum balance is formulated as a first order system and coupled in a nonlinear way to the non-smooth evolution equation of the phase-field variable. We introduce a full discretization in time and space, using a discontinuous Galerkin method for the first order system. Based on this, we show the existence of discrete solutions. We discuss their convergence to a suitable notion of weak solution of the system as the step size in space and time tends to zero and give a comparison to other formulations existing in literature. Simulation results are presented.

This is joint work with Sven Tornquist (Berlin) and Christian Wieners (Karlsruhe) and also based upon collaboration with Kerstin Weinberg and Kai Partmann (both Siegen) within the priority programme "Variational Methods for Predicting Complex Phenomena in Engineering Structures and Materials" (SPP 2256), project "Nonlinear Fracture Dynamics: Modeling, Analysis, Approximation, and Applications", financially supported by the German Research Foundation (DFG).

Contributed talks

Samira Boddin (Universität Kassel)

AM $\neq$ BV

In the engineering community alternate minimization (AM) is a common strategy to calculate minimizers of separately convex energies. It is for instance often applied to rate-independent phase-field damage models, where the energy is even separately quadratic.

From a mathematical point of view we face the difficulty that solutions of rate-independent systems with nonconvex energies might show a discontinuous evolution. In this regard several, in general not equivalent, solution concepts have been developed.

Here the focus lies on balanced viscosity (BV) solutions. They are based on local minimization, thus respect energy barriers, and seem to behave physically reasonable. In [BRKM] a new scheme to approximate BV solutions of a damage model was developed. It combines AM with local minimization and a convergence proof is provided.

Most application related numerical examples suggest that solutions generated by the pure AM scheme converge to BV solutions as well. A first convergence analysis for the pure AM scheme was carried out for the damage model in [KN]. However, it still did not answer the question, if AM solutions are indeed BV solutions. The characterizations of the two types of solutions were similar but not identical.

To answer the question and justify the new scheme with its additional computational cost, we present some finite dimensional examples where the pure AM scheme does not give BV solutions.

[BRKM] S. Boddin, F. Rörentrop, D. Knees, J. Mosler. Approximation of balanced viscosity solutions of a rate-independent damage model by combining alternate minimization with a local minimization algorithm. *arXiv:2211.12940*, 2022.

[KN] D. Knees, M. Negri. Convergence of alternate minimization schemes for phase field fracture and damage. Mathematical Models and Methods in Applied Sciences (M3AS), vol. 27(9), pp. 1743-1794, 2017.

Daniel Böhme (Universität Regensburg)

The two-phase periodic Stokes flow in the plane

We study the two-phase Stokes flow driven by surface tension under the assumption that the sharp interface that separates the fluids (which have equal viscosity) is a periodic graph. The flow is two-dimensional and the fluids fill the entire plane. We first introduce the fundamental solution of the fixed time problem, construct the hydrodynamic single-layer potential and then analyse some related nonlinear (singular) integral operators which are the building blocks of the underlying evolution equation.

Pedro Campos (Universidade de Lisboa)

Unilateral Problems for Quasilinear Operators with Fractional Riesz Gradients

In this work, we extend the classical theory of monotone and pseudomonotone operators to a class of convex constrained problems involving fractional Riesz gradients in bounded and in unbounded domains $\Omega \subset \mathbb{R}^d$:

Find $u \in K$, such that,

$$\int_{\mathbb{R}^d} a(x, u, D^s u) \cdot D^s(v-u)\, dx + \int_\Omega b(x,u)(v-u)\, dx \geq \langle L, v-u \rangle \qquad (1)$$

for all $v \in K$, where $K \subset \Lambda_0^{s,p}(\Omega)$ is a non-empty, closed and convex set of a fractional Sobolev type space $\Lambda_0^{s,p}(\Omega)$ with $0 \leq s \leq 1$ and $1 < p < \infty$, and D^s is the Riesz fractional gradient (for short, fractional gradient) as defined in [5, 6], with $D^1 = D$ is the classical gradient and $\Lambda_0^{1,p}(\Omega) = W_0^{1,p}(\Omega)$, and $D^0 = -R$ is the vector-valued Riesz transform with $\Lambda_0^{0,p}(\Omega) = \{u \in L^p(\mathbb{R}^d) : u = 0 \text{ a.e. in } \Omega\}$.

We discuss the existence, uniqueness and continuous dependence of variational solutions to (1) with respect to the fractional parameter s. We extend the Mosco convergence for convex sets with respect to the parameter s, including the limit cases $s = 1$ and $s = 0$.

We present examples of unilateral problems, including quasi-variational inequalities with constraints of obstacle type and s-gradient type.

Authors: Pedro Campos, José Francisco Rodrigues.

[1] Bellido, J.C., Cueto, J. and Mora-Corral, C., "Γ-convergence of polyconvex functionals involving s-fractional gradients to their local counterparts". Calc. Var. Partial Differential Equations **60**, 7 (2021).

[2] Bruè E., Calzi M., Comi G. E. and Stefani G., "A distributional approach to fractional Sobolev spaces and fractional variation: asymptotics II," C. R. Math. Acad. Sci. Paris, **360**, 589–626 (2022).

[3] Campos, P., "Lions-Calderón spaces and applications to nonlinear fractional partial differential equations" MSc Thesis (2021): `http://hdl.handle.net/10451/52753`.

[4] Lo, C.W.K. and Rodrigues, J.F., "On a class of nonlocal obstacle type problems related to the distributional Riesz fractional derivative" Port. Math. **80**, no. 1/2, 157–205 (2023).

[5] Shieh, T.-T. and Spector, D., "On a new class of fractional partial differential equations," Adv. Calc. Var., **8**, No. 4, 321–336 (2015).

[6] Šilhavý M., "Fractional vector analysis based on invariance requirements (critique of coordinate approaches)", Contin. Mech. Thermodyn., **32**, No. 1, 207–228 (2020).

Jakob Fuchs (Universität Dortmund)

Sharp Interface Reduction of a Mesoscale Model for Two-Species Surfactant Films

We propose a variational model for two-phase surfactant films seperating aqueous and oily fluids. Considering two species of surfactant molecules we describe a phase seperation within the film. The analysis builds on a model for single species lipid biomembranes proposed by Peletier and Röger [ARMA 2009]. We prove a Gamma-convergence result in the limit of vanishing surfactant length and show that the limit inherits a phase seperation and a bending energy.

Joint work with Matthias Röger.

Georg Heinze (Universität Augsburg)

Graph-to-local limit for the nonlocal interaction equation

In this talk I will discuss a class of nonlocal partial differential equations defined in a tensor-weighted space that arise asymptotically from upwind-induced nonlocal dynamics on localising infinite graphs. The convergence of solutions of the graph equations to a solution of the corresponding local equation is achieved with variational methods, exploiting the equations gradient flow nature. The graph gradient structures are nonsymmetric, thus leading to Finsler-type graph gradient flows. In contrast, the limiting local gradient flow corresponds to a symmetric Otto-Wasserstein gradient structure, so that the presented graph-to-local limit is in fact also a non-symmetric-to-symmetric limit.

Richard Höfer (Universität Regensburg)

Hydrodynamic limit of multiscale viscoelastic models for suspensions of rigid Brownian particles

It is well-known that particulate flows display non-Newtonian, viscoelastic behavior. On the one hand, microscopic considerations lead to multiscale models coupling a Vlasov-Fokker-Planck equation for the dispersed phase to the fluid equations. On the other hand, many macroscopic models have been proposed for the modeling of viscoelastic fluids, and a natural questions how multiscale and macroscopic equations are related to each other.

We study the multiscale Doi(-Saintillan-Shelley) model for (active) Brownian rigid rod-like suspensions. We consider the regime of a small Weissenberg number, which corresponds to a fast rotational diffusion compared to the fluid velocity gradient, and we analyze the resulting hydrodynamic approximation: we show the asymptotic validity of macroscopic ordered fluid models with stress diffusion as expansion in the Weissenberg number. The result holds for zero Reynolds number in 3D and for general Reynolds number in 2D. This is joint work with Mitia Duerinckx, Lucas Ertzbischoff and Alexandre Girodroux-Lavigne.

Christoph Hurm (Universität Regensburg)

Strong Nonlocal-to-Local Convergence of the Cahn-Hilliard Equation and its Operator

We prove convergence of a sequence of weak solutions of the nonlocal Cahn-Hilliard equation to the strong solution of the corresponding local Cahn-Hilliard equation. The analysis is done in the case of sufficiently smooth bounded domains with Neumann boundary condition and a $W^{1,1}$-kernel.

The proof is based on the relative entropy method. Additionally, we prove the strong L^2-convergence of the nonlocal operator to the negative Laplacian together with a rate of convergence.

Patrik Knopf (Universität Regensburg)

The anisotropic Cahn–Hilliard equation: regularity theory and strict separation properties

The Cahn–Hilliard equation with anisotropic energy contributions frequently appears in many physical systems, e.g., related to crystal growth. Compared to the standard isotropic variant, the mathematical analysis of the anisotropic Cahn–Hilliard equation is much more involved. This is because physically relevant anisotropy functions are non-smooth and make the system of PDEs highly nonlinear. We discuss the existence, uniqueness, and regularity of weak solutions. In particular, to obtain higher spatial regularity of weak solutions, we show new regularity results for quasilinear elliptic equations of second order. For the anisotropic Cahn–Hilliard equation with a logarithmic free energy, we further obtain so-called separation properties, which state that the order parameter stays away from the values representing the pure phases.

Authors: Harald Garcke, Patrik Knopf, Julia Wittmann

Lukas Koch (Max-Planck-Institut Leipzig)

Convex duality and regularity

I will present regularity results for minimisers of a new class of non-uniformly elliptic integrands, modeled on polynomials. As a particular highlight, when the integrand is a uniformly convex polynomial, the approach allows to prove local analyticity of minimisers in two space dimensions. This is based on joint work with C. de Filippis and J. Kristensen.

Chiara Leone (Università degli Studi di Napoli Federico II)

$\mathcal{A}$-caloric approximation and partial regularity

In this seminar we discuss a new $\mathcal{A}$-caloric approximation lemma compatible with an Orlicz setting. With this result, we will establish a partial regularity result for parabolic systems of the type

$$u_t - \operatorname{div} a(Du) = 0,$$

where the growth of a is bounded by the derivative of an N-function φ. The primary assumption for φ is that $t\varphi''(t)$ and $\varphi'(t)$ are uniformly comparable on $(0, \infty)$.

The results presented are contained in a paper in collaboration with Mikil Foss, Teresa Isernia and Anna Verde.

Yadong Liu (Universität Regensburg)

On a compressible fluid-structure interaction problem with slip boundary conditions

The talk will consider a compressible barotropic fluid system interacting with a linear (visco)-elastic solid equation. In particular, the elastic structure is part of the fluid boundary and the boundary is taken to move freely. We show the existence of weak solutions to the coupled system provided the adiabatic exponent satisfies $\gamma > \frac{3}{2}$ with structure damping and $\gamma > \frac{12}{7}$ without damping, utilizing the domain extension and regularization approximation. Moreover, via a modified relative entropy method in the time-dependent domain, we prove the weak-strong uniqueness property of weak solutions. Finally, a rigorous justification of the incompressible inviscid limit of the compressible fluid-structure interaction problem will be given, in the regime of low Mach number, high Reynolds number, and well-prepared initial data. This is the first result concerning such singular limits problem for compressible Navier–Stokes equations with an elastic boundary.

Maximilian Moser (Institute of Science and Technology Austria)

Sharp interface limit via relative entropy: Navier-Stokes/Allen Cahn system with vanishing mobility

In this talk we consider the sharp interface limit of a Navier-Stokes/Allen Cahn equation in a bounded smooth domain in two dimensions when the small parameter $\epsilon > 0$ related to the thickness of the diffuse interface is sent to zero. The mobility in the Allen-Cahn equation is scaled in such a way that it converges to zero polynomially and less than quadratic with repect to ϵ. The limit problem is given by the classical two-phase Navier-Stokes-system with surface tension and we show convergence for well-prepared initial data and for small times such that a strong solution to the limit problem exists. The approach is via the relative entropy method, i.e. one shows a Gronwall-type estimate for suitable energy functionals that control the error between the solution to the diffuse and sharp interface systems in a suitable way. This is joint work with H. Abels (Univ. of Regensburg) and J. Fischer (IST Austria).

Timo Neumeier (Universität Augsburg)

Computational polyconvexifcation of isotropic functions

Polyconvex relaxation is a reasonable option when faced with the minimization of energy functionals with non-convex energy densities. For this purpose, we present a method for the determination of the polyconvex envelope of isotropic energy densities, which can be characterized by means of their signed singular value representations. This leads to a simple algorithm for the numerical approximation of the polyconvex envelope. Instead of operating on the d^2-dimensional space of matrices (the deformation gradient in typical applications), the algorithm requires only the computation of the lower convex envelope of a d-dimensional manifold. This task is realized by standard algorithms from computational geometry or an optimization approach. The significant computational speedup, originating from the dimension reduction by moving from full matrix space to the space of signed singular values, is demonstrated in a series of numerical experiments.

Authors: Timo Neumeier, Malte A. Peter, Daniel Peterseim, David Wiedemann

Jan-Frederik Pietschmann (Universität Augsburg)

Optimal transport and gradient flows on metric graphs

We consider a formulation of dynamic optimal transport on metric (or quantum) graphs. We show wellposedness via a duality principle. In a second step, we consider gradient flows on metric graphs and study several limits in terms of time or capacity rescaling. This allows us to recover dynamics in which the vertices directly interact, connecting our framework to discrete graphs.

Richard Schubert (Universität Bonn)

Convergence to the flat geometry for the Mullins-Sekerka evolution

We consider the evolution of fairly wild perturbations of the plane in three space dimensions by the Mullins-Sekerka law. Mullins Sekerka is a non-local third-order geometric evolution that is characterized by preservation of mass and reduction of surface area. Only assuming initial finiteness of the excess mass and the excess surface energy, we prove that the surface eventually becomes a graph, and that the energy converges with an optimal algebraic rate towards the flat ground state. I will discuss how the gradient flow structure and the L^1-method for conservation laws crucially enter the proof. Based on joint work with Felix Otto (Leipzig) and Maria G. Westdickenberg (Aachen).

Jan-Eric Sulzbach (TU München)

Slow Manifolds in fast-reaction PDEs

Multiple-time-scale systems arise in various applications from climate science to bio-chemistry and in recent decades the geometric singular perturbation theory has emerged as a powerful mathematical theory for the analysis of these multi-scale systems in finite dimensions. However, it is very challenging to transfer the results to infinite dimensions. In this talk we focus on fast-reaction systems in general Banach spaces and we extend the classical Fenichel theory to this infinite-dimensional setting. The key steps in this process are to define a new notion of normal hyperbolicity and to find the right function spaces for the convergence of the fast-reaction system to its limit system as the time-scale parameter tends to zero. With this we can show that the solution of the fast-reaction system is approximated by the corresponding slow flow of the limit system. Moreover, introducing an additional parameter that stems from a splitting in the slow variable space, we construct a family of slow manifolds and prove that the slow manifolds are close to the critical manifold. And lastly, we prove that the semi-flow on the slow manifold converges to the semi-flow on the critical manifold. This is joint work with Christian Kuehn (TUM).

Andreas Vikelis (Universität Wien)

Measure-valued solutions for non-associative finite plasticity

The variational treatment of evolutionary nonassociative elasto-plasticity at finite strains remains unexplored. In this direction, following the concept of energetic solutions, we present an existence result for measure-valued solutions of the quasistatic evolution problem which are stable and balance the energy. In particular, we apply a modification of the standard time-discretization scheme, considering Young measures generated by piecewise constant interpolants of time-discrete solutions of a properly defined minimization problem. A key point in our analysis is the limit passage in the dissipation. The later calls for time-continuity properties of the stresses which are not expected in the quasistatic framework. To overcome this obstacle we introduce a regularization of the generalized stress in the definition of our energetic solutions. Joint work with Ulisse Stefanelli.