> A major task of mathematics today is to harmonize the continuous and the discrete, to include them in one comprehensive mathematics, and to eliminate obscurity from both.
>
> *E.T. Bell, "Men of Mathematics", Simon and Schuster, New York.*

This task is fulfilled by the so called

Calculus on Measure Chains

which was founded in my doctoral dissertation in 1988. It allows to unify the usual differential and integral calculus for one variable. One can replace the range of definition ($\mathbb{R}$) of the functions under consideration by an arbitrary MEASURE CHAIN (or TIME SCALE) $\mathbb{T}$.

Each closed subset $\mathbb{T}$ of $\mathbb{R}$ bears the structure of a MEASURE CHAIN in a natural way.

The reals $\mathbb{Z}$	Ordinary Calculus
The integer numbers $\mathbb{Z}$	
Integer sequence with constant distance $h\mathbb{Z}$	Ordinary Difference Calculus with h
Discrete point sequence	Difference Calculus with variable step size
Sequence of closed intervals	Differential Calculus with pulses
$q^{\mathbb{Z}}$, $q > 1$	q–difference equations
Cantor set $\mathbb{D}$	exotic time scale, the theory also works

The derivative for functions $f : \mathbb{T} \to X$, (X Banach space) is defined by

$$f^{\Delta}(t) := \lim_{\substack{s \to t \\ s \in \mathbb{T}}} \frac{f(\sigma(t)) - f(s)}{\sigma(t) - s},$$

where $\sigma(t) := \inf\{r \in \mathbb{T} \big| r > t\}$ and the limit has to be taken with respect to the relative topology of $\mathbb{T}$.

Based on this notion one can develop wide areas of ordinary calculus such as

- Basic algebraic properties of the derivative, mean value theorem,
- Special (exponential, trigonometric, hyperbolic) functions and their properties.
- Theorems on existence, uniqueness and parameter dependence for ordinary diff equations,
- Linear theory, generalized matrix exponential function and its properties
- Tools for qualitative theory of dynamical systems: Gronwall lemma, invariant manifolds, topological equivalence,
- Sturm–Liouville Theory: separation theorem, comparison theorem oscillation theorems, Riccati transformations, symplectic systems, Picone identity
- Laplace/z–transform, Fourier transform,
- Discrete analogs of Heisenberg's uncertainty relation, heat equation, Schrödinger harmonic oscillator.

It has been confirmed many times: Whenever the transition $h \to 0$ is subject to mathematical considerations the usual heuristic procedure of arguing via analogy or step by step translation of one case into the other can be replaced by this closed mathematical concept.

Stefan Hilger

Starting in 1997 calculus on TIME SCALES has received considerable international attention, especially in the United States (See. `http://www.math.unl.edu/ apeterso/timescaleaut.html`). Since then about 300 journal articles and three books [V6 – V8] about TIME SCALES have been published. It has become the framework for several PhD or habilitation ([V4]) theses. Several workshops and special sessions on TIME SCALES were organized. In July 2003 calculus on TIME SCALES was topic of a cover story in the popular journal "The New Scientist" [V1].

Some References

V1 Vanessa Spedding, Taming nature's numbers. *New Scientist – The global science and technology weekly*, 179 (2003), S. 28 – 31.

V2 Christian Pötzsche. *Langsame Faserbündel dynamischer Gleichungen auf Maßketten*. PhD thesis, Universität Augsburg, Augsburg, 2002.

V3 Stefan Keller. *Asymptotisches Verhalten invarianter Faserbündel bei Diskretisierung und Mittelwertbildung im Rahmen der Analysis auf Zeitskalen*. PhD thesis, Universität Augsburg, Augsburg, 2000.

V4 Roman Hilscher. *Quadratic Functionals in Discrete optimal Control*. 2002. Habilitation Thesis, Masaryk University Brno.

V5 Roman Hilscher. *Some Aspects of the Qualitative Theory of Linear Hamiltonian Systems*. PhD thesis, Masaryk University Brno, 1998.

V6 Billûr Kaymakçalan, V. Lakshmikantham, and S. Sivasundaram. *Dynamic Systems on Measure Chains*, volume 370 of *Mathematics and its Applications*. Kluwer Academic Publishers, Dordrecht, 1996.

V7 Martin Bohner and Allan Peterson, editors. *Advances in Dynamic Equations on Time Scales*. Birkhäuser, Boston, 2003.

V8 Martin Bohner and Allan C. Peterson. *Dynamic Equations on Time Scales — An Introduction with Applications*. Birkhäuser, Boston, Basel, Berlin, 2001.

V9 Special issue entitled "Dynamic Equations on Time Scales" of the Journal of Computational and Applied Mathematics, Vol. 141, 1 – 2, pp. 1 – 284 (2002).

Workshops and congress sections

- First International Workshop on Dynamic Equations on Time Scales, Istanbul, Turkey, June 27 – July 2, 2005.
- Special session on "Time Scales" at the Fourth World Congress of Nonlinear Analysts WCNA–2004, Orlando, Florida USA, June 30 – July 7, 2004.
- Special session on "Time Scales" at the Joint Mathematics Meeting (993) in Phoenix, AZ, January 7–10, 2004.
- Special session Workshop on Time Scales and Applications at the Fourth International Conference on Dynamic Systems and Applications in Atlanta, GA, May 21–24, 2003.
- Time Scales Workshop, Dayton, Ohio, September 20–21, 2002.
- Rocky Mountain Mathematics Consortium Summer Conference on "Dynamic Equations on Time Scales and Their Applications" July 8–19, 2002, University of Wyoming, Laramie, Wyoming.
- Special session on "Dynamic Equations on Time Scales" at the Spring Southeastern Section Meeting (975) in Atlanta, GA, March 8–10, 2002.
- Special session on "Dynamic Equations on Time Scales" at the Joint Mathematics Meeting (973) in San Diego, CA, January 6–9, 2002.

- Special session on "Dynamic Equations on Time Scales" at the SIAM SEAS Annual Conference at Coastal Carolina University in Myrtle Beach, SC, March 16–17, 2001.
- Fargo Preconference Workshop on BVPs and Oscillation Theory of Differential Equations on Measure Chains, Fargo, North Dakota, October 19, 2000.
- Special session on "Time Scales" at the Third World Congress of Nonlinear Analysts WCNA–2000 in Catania, Sicily, Italy, July 19-26, 2000.