6. Dynamic Utility Functions

Objectives:

Understanding of

- Time separable utility functions
- CRRA time separable preferences
- Prospect theory
- Recursive preferences

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6.1 Time-Separable Utility Function

The inter-temporal behavior will in many financial models be presented by a time separable (additive) inter-temporal utility function with a constant relative risk aversion and infinite time horizon. . This utility function is not only a standard in Financial Theory but also in modern Dynamic Macroeconomics.

In most applications the utility function is defined on consumption streams rather than on (stocks of) wealth. In order to understand the connection we keep in mind the inter-temporal budget restriction in form of a simple difference equation

$$W_t = \left(1 + R_t\right)W_{t-1} - c_t \tag{6.1}$$

The time separable utility function can be written as:

$$E\left[U_1\left(c_1,c_2,\ldots,c_t,\ldots\right)\right] = E_1\left[\sum_{t=1}^{\infty}\beta^{t-1}u\left(c_t\right)\right] = u\left(c_1\right) + E_1\left[\beta U_2\left(c_2,c_3,\ldots,c_t,\ldots\right)\right] \tag{6.2}$$

In many models we use a CRRA specification of the utility function:

$$E\left[U\left(c_1,c_2,\ldots,c_t,\ldots\right)\right] = E_1\left[\sum_{t=1}^{\infty}\beta^{t-1}\frac{c_t^{1-\gamma}}{1-\gamma}\right] = \frac{c_1^{1-\gamma}}{1-\gamma} + E_1\left[\beta\frac{c_2^{1-\gamma}}{1-\gamma} + \ldots + \beta^{t-1}\frac{c_t^{1-\gamma}}{1-\gamma}\ldots\right] \tag{6.3}$$

In order to understand the central aspects of the inter-temporal behavior it is convenient to inspect three important inter-temporal properties of this utility function:

- Relative Risk Aversion (CRRA)
- Inter-temporal marginal rate of substitution (IMRS)
- Inter-temporal elasticity of substitution (IES)

6.1.1 Relative Risk Aversion

The relative risk aversion (with respect to consumption is defined as

$$RRA \equiv -\frac{u''\left(c_t\right)}{u'\left(c_t\right)}c_t\,. \tag{6.4}$$

The RRA can be understood as the elasticity of the marginal utility with respect to consumption. In general this elasticity varies with the consumption level.

In the CRRA-case of the time separable utility function the RRA equals

$$RRA = -\frac{-\gamma c_t^{-\gamma-1}}{c_t^{-\gamma}}c_t = \gamma\,. \tag{6.5}$$

RRA is constant and therefore independent from the level of consumption.

6.1.2 Inter-temporal Marginal Rate of Substitution

IMRS defines the quantity of the good available in *t+1* that will be exchanged for a small unit of the good available in *t*. If we use the additive inter-temporal utility function for any pair of consumption flow of the periods *t* and *t+1* the IMRS is defined as:

$$dU = E_1\left[\beta^{t-1}u'\left(c_t\right)\right]dc_t + E_1\left[\beta^{t}u'\left(c_{t+1}\right)\right]dc_{t+1} = 0 \qquad (6.6)$$

From this follows the inter-temporal marginal rate of substitution:

$$IMRS_{t+1/t} \equiv \frac{dc_{t+1}}{dc_t} = -E_1\left[\frac{1}{\beta}\frac{u'\left(c_t\right)}{u'\left(c_{t+1}\right)}\right] \qquad (6.7)$$

Obviously the IMRS (of consumption of two periods *t* and *t+1*) of the time separable inter-temporal utility function is equal to:

$$IMRS_{t+1/t} = -E_1\left[\frac{1}{\beta}\left(\frac{c_t}{c_{t+1}}\right)^{-\gamma}\right] = -E_1\left[\frac{1}{\beta}\left(\frac{c_{t+1}}{c_t}\right)^{\gamma}\right] \qquad (6.8)$$

6.1.3 Inter-temporal Elasticity of Substitution

The IES explains the relative change of the fraction of consumption streams of two ongoing periods in reaction to a relative change of the IMRS

$$IES_{t+1/t} \equiv \frac{d\frac{c_{t+1}}{c_t}\Big/\frac{c_{t+1}}{c_t}}{dIMRS_{t+1/t}\big/IMRS_{t+1/t}} = \frac{d\frac{c_{t+1}}{c_t}\,IMRS_{t+1/t}}{dIMRS_{t+1/t}\,\frac{c_{t+1}}{c_t}} \qquad (6.9)$$

According to this definition the IES of a CRRA utility function can be calculated as

$$IES_{t+1/t} = \frac{d\frac{c_{t+1}}{c_t}E_1\left[\frac{1}{\beta}\left(\frac{c_{t+1}}{c_t}\right)^{\gamma}\right]}{dIMRS_{t+1/t}\,\frac{c_{t+1}}{c_t}} = \frac{E_1\left[\frac{1}{\beta}\left(\frac{c_{t+1}}{c_t}\right)^{\gamma-1}\right]}{\frac{dIMRS_{t+1/t}}{d\left(c_{t+1}/c_t\right)}} = \frac{E_1\left[\frac{1}{\beta}\left(\frac{c_{t+1}}{c_t}\right)^{\gamma-1}\right]}{\gamma E_1\left[\frac{1}{\beta}\left(\frac{c_{t+1}}{c_t}\right)^{\gamma-1}\right]} = \frac{1}{\gamma}. \qquad (6.10)$$

Obviously the IES is equal to the inverse of the RRA. This relation is one reason for the pretended attractiveness of a time separable utility function. A-temporal and inter-temporal features interact in a simple way.

The CRRA assumption is a wonderful didactical simplification, but more and more it turns out to be an oversimplification for empirical work in modern macroeconomics and finance. To overcome the latter various forms of generalized utility function has been developed.

6.2 Critique of the Expected Utility Hypothesis

Maurice F. M. Allais was the protagonist of the experimental analysis. He proposed a game with the pay-offs and the utilities

$$\left\{x_1,x_2,x_3\right\} = \left\{0€;\ 1{,}000{,}000€;\ 5{,}000{,}000€\right\}, \text{ and} \qquad (6.11a)$$

$$u\left(x_1\right) < u\left(x_2\right) < u\left(x_3\right). \qquad (6.11b)$$

As a first choice (step *I*) he offered two alternatives *A, B*:

$$l_I^A = (0;\ 1;\ 0) \qquad\qquad l_I^B = (0.89;\ 0.11;0) \tag{6.12}$$

Moreover, as a second choice (step *II*) he offered again two alternative lotteries *A, B*:

$$l_{II}^A = (0.01;\ 0.89;\ 0.1) \qquad\qquad l_{II}^B = (0.9;\ 0;\ 0.1) \tag{6.13}$$

Empirical experiments proofed that many participants of the experiments owe the following preferences about lotteries:

$$l_I^A \succ l_{II}^A \quad \wedge \quad l_{II}^B \succ l_I^B \tag{6.14}$$

According to the Expected Utility Hypothesis the preferences of the individuals can be represented by the following utility function:

$$U = \pi_1 u(x_1) + \pi_2 u(x_2) + \pi_3 u(x_3) \tag{6.15}$$

In order to present these preferences about different lotteries over the payoffs of the games given above in a two-dimensional graph it is convenient to rewrite the expected utility as

$$U = \pi_1 u(x_1) + (1 - \pi_1 - \pi_3) u(x_2) + \pi_3 u(x_3) \tag{6.16}$$

Any indifference curve among different lotteries has to satisfy the condition

$$dU = d\pi_1 u(x_1) + (1 - d\pi_1 - d\pi_3) u(x_2) + d\pi_3 u(x_3) = 0\,. \tag{6.17}$$

Thus, the slope of the indifference curve is thus determined as

$$\frac{d\pi_3}{d\pi_1} = -\frac{u(x_1) - u(x_2)}{u(x_3) - u(x_2)} > 0\,. \tag{6.18}$$

Given the payoffs of the lotteries the slope of the indifference curve is positive, since the numerator (denominator) of the fraction is negative (positive). Moreover the slope depends only on the on the utility of the payoffs, and is thus independent from their probabilities.

The indifference curves can be presented in the so-called Marshak-Matchina-Triangle.

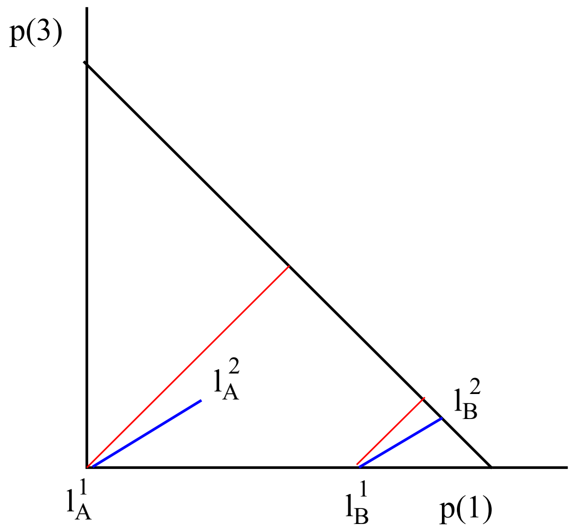

Figure 6.1: Marshak-Matchina-Triangle

The lowest utility level is associated to the point $p(1)=1$; the highest to the point $p(3)=1$. Whether the individuals prefer the lotteries of type I, or II depends on the slope of the indifference curves. If the slope of the indifference curve coincides with the slope of the red lines, the individual is indifferent between the two types of lotteries. If the slopes of the (blue) indifference curves are steeper than the (red) reference line, the individual prefers lotteries of type I over lotteries of type II. If on the contrary, the slope of the (green) indifference curves is more flat then the (red) reference line the lotteries of type II are preferred over lotteries of type I. However, preferences as in (6.14) are inconsistent with the Expected Utility Hypothesis.