On the Problem of Poreda

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Abstract. We investigate a problem posed by Poreda on the behaviour of the strong uniqueness constant with increased polynomial degree. It has been conjectured by Bartelt and McLaughlin that this constants tends to zero for all non-polynomial functions. In this paper, we give evidence for this and prove a special result, which we conjecture to be a worst case result.

§1 Introduction to the Problem

Let us first introduce the necessary notations. We donote by

$$e_n(f) = d(f, \mathcal{P}_n) := \min_{p_n \in \mathcal{P}_n} \|f - p_n\|.$$

the error of best approximation with respect to the space $\mathcal{P}_n$ of polynomials of degree n. The norm being the sup-norm

$$\|g\| = \|g\|_{[a,b]} := \sup_{x \in [a,b]} |f(x)|$$

and denote the best approximation by $P_n^* f$; i.e.,

$$\|f - P_n^* f\| = e_n(f),$$

Then, Freud proved, that for some smallest constant $L_n(f)$

$$\frac{\|P_n^* f - P_n^* g\|}{\|f - g\|} \leq L_n(f)$$

and due to Newman and Shapiro [9] for some largest constant $\gamma_n(f)$

$$\|f - q_n\| \geq \|f - p_n^*(f)\| + \gamma_n(f) \|q_n - P_n^* f\|, \qquad \text{for all } q_n \in \mathcal{P}_n.$$

Furthermore

$$L_n(f) \leq \frac{2}{\gamma_n(f)}.$$

The proof of this and other details may be found in the book of Cheney [3].

Poreda [8] asked how the strong uniquness constant behaves as $n \to \infty$. Bartet and McLaughlin [2] conjectured that

$$\liminf_{n\to\infty} \gamma_n(f) = 0$$

for all non-polynomial f.

We do not have enough space here to cover all results, which throw light on this conjecture. The simplest and yet most useful estimate is due to Blatt [1]. He proved that

$$\gamma_n(f) \le \frac{1}{n+1},$$

if $f - p_n^*(f)$ has exactly $n+2$ alternation points. This can be extended a little bit to extramal error sets with slightly more points (see [7]). Moreover, using a real Carathéodory-Fejér method of Gutknecht and Trefethen one can use this result to prove that the conjecture is true for functions, which are analytic in a certain neighborhood of $[-1,1]$ (see [5] and [6]).

We will need the characterization of Bartelt and Mclaughlin

$$\begin{aligned}\gamma_n(f) \quad = \quad & \inf\{h \in \mathbb{R} : \text{There is a } p \in \mathcal{P}_n, \\ & \qquad \|p\| = 1 \text{ with } s_n f(x) p(x) \le h \text{ for } x \in E_n(f)\}.\end{aligned}$$

where $s_n f(x) = \text{sign } (f - p_n^*)(x)$ and $E_n(f)$ is the extremal set of the error function $f - p_n^*$. For a subsequence, $E_n(f)$ will consist of $n+2$ subsets $E_0 < \ldots < E_{n+1}$, such that the error function oscillates in sign on these subsets. We choose points

$$x_0 = -1 \le E_0 < x_1 < E_2 < \ldots < E_{n+1} \le x_{n+2} = 1$$

and fix a function σ_n with

$$\sigma_n(x) = (-1)^i, \qquad x \in (x_i, x_{i+1}).$$

Then 0 is the best approximation of σ_n with respect to $\mathcal{P}_n$ and

$$\gamma_n(f) \le \gamma_n(\sigma_n)$$

by the chracterization above. Thus we only investigate such extremal functions. We will call such a function a signum function of order $n+2$.

Conjecture 1.

$$\gamma_n(f) \le \frac{C}{\log n}$$

for some constant $C > 0$ independend of n and f.

Strong uniqueness does hold for approximation in the space $C_{2\pi}$ of 2π-periodic continuous functions with respect to $\mathcal{T}_n$, the set of all trigonometric polynomials of degree n. We denote the corresponding strong uniqueness constant by $\hat{\gamma}(f)$.

Let us explain, why it suffices to prove the conjecture for the trigonometric case. For functions $g : [-1, 1] \to \mathbb{R}$ we set

$$\hat{g}(t) = g(\cos t), \qquad t \in [0, 2\pi).$$

We start with a signum function σ_n of order $n+2$ on $[-1, 1]$. Then the best approximation to $\hat{\sigma}_n$ with respect to $\mathcal{T}_n$ on $[0, 2\pi)$ is 0, because $\hat{\sigma}_n$ has $2n+2$ alternating extremal points. By the characterization above, we find a function $v \in \mathcal{T}_n$ with $\|v\| = 1$ and

$$v(t)\hat{\sigma}_n(t) \leq \hat{\gamma}(\hat{\sigma}_n).$$

Take $w(t) = (v(t) + v(-t))/2$, and $p \in \mathcal{P}_n$ such that $\hat{p} = w$. Then

$$\|p\| \geq \frac{1 - \hat{\gamma}(\hat{\sigma}_n)}{2},$$

and

$$p(x)\sigma_n(x) \leq \hat{\gamma}(\hat{\sigma}_n).$$

We get the following Lemma

Lemma 1. *For any sign function σ_n on $[-1, 1]$ of order n, we get*

$$\gamma_n(\sigma_n) \leq \frac{2\hat{\gamma}(\hat{\sigma}_n)}{1 - \hat{\gamma}(\hat{\sigma}_n)}.$$

§2 Results

The following result takes care of an extremal case.

Theorem 1. *Let s_n be the trigonometric sign function, defined as*

$$s_n(t) = (-1)^\nu, \qquad t \in \left[\frac{\nu\pi}{n+1}, \frac{(\nu+1)\pi}{n+1}\right)$$

for all $t \in [0, 2\pi)$. Then

$$\frac{C_1}{\log n} \leq \hat{\gamma}_n(s_n) \leq \frac{C_2}{\log n}$$

for constants $0 < C_1 < C_2$.

The rest of this section is devoted to the proof of this Theorem.

Let $v_n \in \mathcal{T}_n$ be the trigonometric polynomial of maximal norm, such that

$$v_n(t)s_n(t) \leq 1$$

for all t. We may assume that this polyomial takes its norm at $t_0 \in [0, \pi/(n+1)]$, where it is negative. Then the polynomial will maximize $v(t_0)$ under the restriction $v_n(t)s_n(t) \leq 1$. By the theory of semi-infinite optimization, there must be points $t_\nu \in [\nu\pi/(n+1), (\nu+1)\pi/(n+1)]$ for $\nu = 1, \ldots, 2n+1$ such that $v_n(t_\nu) = (-1)^\nu$. A simple zero counting argument shows, that this maximization problem has a unique solution, which is independend of the point t_0. A symmetry argument shows that it is symmetric to $\pi/(2n+2)$, where it takes its norm.

We will now investigate

$$w_n(t) = -v_n(t + \pi/(2n+2)), \qquad t \in [-\pi, \pi].$$

This is an even polynomial and takes its norm in 0. It maximizes $w_n(0)$ among all polynomials with

$$(-1)^\nu w_n(t) \geq -1, \qquad t \in I_\nu := \left[\frac{\nu\pi}{2n+2}, \frac{(\nu+2)\pi}{2n+2}\right], \quad \nu \text{ odd}.$$

We will study w_n on $[0, \pi]$. On this interval, it is in $\mathcal{C}_n$, the space of all cosine polynomials of maximal degree n. We compare it to the polynomial $v_n \in \mathcal{C}_n$ defined by the $n+1$ interpolation properties

$$v_n(\tau_\nu) = (-1)^{\frac{\nu-1}{2}}, \qquad \tau_\nu = \frac{\nu\pi}{2n+1}, \quad \nu = 1, 3, 5, \ldots, 2n+1. \quad (2.1)$$

Note $\tau_\nu \in I_\nu$. By a simple zero counting estimate, we have

$$v_n(0) > w_n(0).$$

This will yield an upper estimate for w_n and thus a lower estimate for $\hat{\gamma}(s_n)$.

We mention that we could now use well known results about the best interpolation norm in the trigonometric case. But we need more detailed information anyway.

Consider the polynomial $\phi_n \in \mathcal{T}_n$ defined by

$$\phi_n(t) = \operatorname{Re} \frac{z^{2n+1} - 1}{(2n+1)z^n(z-1)} = \operatorname{Re} \frac{z^{-n} + \ldots + z^n}{2n+1}.$$

with $z = e^{it}$. Setting

$$\phi_{\nu,n}(z) = \phi_n(z - \tau_\nu) + \phi_n(z + \tau_\nu),$$

we see, that we have just constructed the Lagrange polynomials for the interpolation problem (2.1). To estimate $|\phi_{\nu,n}(0)|$, we need need to compute ϕ_n at odd multiples of $d_n := \pi/(2n+1)$. Setting

$$z_{\nu,n} = e^{i\nu d_n}$$

we get for ν odd

$$|\phi_n(\nu d_n)| = \frac{2}{2n+1}\left|\mathrm{Re}\,\left(\frac{z_{\nu,n}^n}{z_{\nu,n}-1}\right)\right| \leq \frac{1}{(2n+1)|z_{\nu,n}-1|}.$$

Since

$$|z_{\nu,n}-1| \geq \frac{|\nu d_n|}{2},$$

we get

$$|\phi_n(\nu d_n)| \leq \frac{2}{\nu\pi}.$$

Thus

$$|\phi_{\nu,n}(0)| = 2|\phi_n(\tau_\nu)| \leq \frac{4}{\nu\pi}.$$

Summing up the Lagrange polynomials, we get

$$\begin{aligned} w_n(0) &\leq v_n(0) \\ &= |\phi_{1,n}(0)| + |\phi_{3,n}(0)| + \ldots + |\phi_{2n+1,n}(0)| \\ &\leq D_1 \log(n) \end{aligned}$$

with a constant $D_1 > 0$, which does not depend on n.

We can also derive a lower estimate for $v_n(0)$. Since $z_{\nu,n}^n \to i$ for $n \to \infty$, we get for n large enough and $|\nu d_n| \leq \pi/2$

$$\begin{aligned} |\phi_n(\nu d_n)| &\geq \frac{\mathrm{Im}\,(z_{\nu,n}-1)}{(2n+1)|z_{\nu,n}-1|^2} \\ &\geq \frac{1}{2(2n+1)|\sin(\nu d_n)|} \\ &\geq \frac{1}{2(2n+1)|\nu d_n|} \\ &= \frac{1}{2\pi\nu}. \end{aligned}$$

Thus we see

$$v_n(0) \geq D_2 \log(n),$$

and our estimate for $v_n(0)$ was sharp.

To get a lower bound for $w_n(0)$, we really need to construct a polynomial $h_n \in C_n$, which satisfies the conditions

$$(-1)^\nu h_n(t) \leq 1 \tag{2.2}$$

for

$$t \in [(2\nu+1)\pi/(2n+2), (2\nu+3)\pi/(2n+2)] \cap [0,\pi].$$

for $\nu = -1, \dots, n$. $h_n(0)$ will then be a lower estimate for $w_n(0)$.

We could use the polyonomial v_n, properly scaled, for this purpose, but the proofs get very technical. Thus we use instead the polyomials

$$u_n(t) = \sum_{\nu=1}^{n_\alpha} \phi_{\nu,n}(t),$$

with $n_\alpha = [n^\alpha]$ (using the Gaussian bracket) for some $0 < \alpha < 1$ and n big enough. Then with the same estimates as above, we still get

$$u_n(0) \geq D_3 \log(n)$$

for some constant $D_3 > 0$. Morevover, if we take some $0 < \alpha < \beta < 1$, we have by the representation of ϕ_n

$$|u_n(t)| \leq D_4 n^{\alpha-\beta}, \qquad \frac{\pi}{n^\beta} \leq |t| \leq \pi.$$

Clearly

$$u_n(t) \geq 1, \qquad t \in [0, \pi/(2n+2)].$$

By elementary calculations, we see that absolute values of the derivatives of ϕ_n taken in the zeros of ϕ_n

$$s_\nu = \frac{2\nu\pi}{2n+1}, \qquad \nu \neq 0$$

are monotonically decreasing. Using this, we get that

$$u_n'(\tau_1) < 0, \quad u_n'(\tau_2) > 0, \dots$$

With a zero counting argument applied to u_n', we see that we only need to estimate u_ν in the intervals

$$[\nu\pi/(2n+2), \nu\pi/(2n+1)]$$

for odd ν with $\nu \leq n^\beta$. On the right endpoint of this interval, we have values ± 1 or 0 by the interpolation property. Repeating the same estimate as above, we can prove

$$\|u_n\|_{[0,\pi]} \leq D_5 \log n$$

for some constant $D_5 > 0$. By the Markov inequality, we finally see that $u_n/2$ for big enough n satisfies the inequalities (2.2).

This finishes the proof of the Theorem.

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